

Probabilistic Process Algebra to Unifying Quantum and Classical Computing in Closed Systems

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Abstract. We have unified quantum and classical computing in open quantum systems called qACP which is a quantum generalization of process algebra ACP. But, an axiomatization for quantum and classical processes with an assumption of closed quantum systems is still missing. For closed quantum systems, unitary operator, quantum measurement and quantum entanglement are three basic components for quantum computing. This leads to probability unavoidable. Along the solution of qACP to unify quantum and classical computing in open quantum systems, we unify quantum and classical computing with an assumption of closed systems under the framework of ACP-like probabilistic process algebra. This unification make it can be used widely in verification for quantum and classical computing mixed systems, such as most quantum communication protocols.

Keywords: Quantum Processes; Probabilistic Process Algebra; Algebra of Communicating Processes; Axiomatization

1. Introduction

Quantum process algebra provides formal tools for modeling, analysis and verification of quantum communication protocols, which combine quantum communications and quantum computing together. [7] [8] defined a language called CQP (Communicating Quantum Processes) by adding primitives for quantum measurements and transformation of quantum states to π -calculus. An operational semantics and a type system for CQP were also presented to prove that the semantics preserves typing and typing guarantees that each qubit is owned by a unique process within a system.

[9] [10] [12] [11] defined a language called QPA_{lg} (Quantum Process Algebra), in which, based on CCS

[2], primitives of unitary transformations and quantum measurements were added to CCS. An operational semantics based on probabilistic branching bisimulation was given in QPAlg.

qCCS [13] was introduced as a kind of algebra of pure quantum processes (no classical data involved) based on CCS. qCCS aimed at providing a suitable framework, in which the mechanism of quantum concurrent computation can be understood, and interactions and conjugation of computation and communication in quantum systems can be observed. In qCCS, quantum operations (super operators) were chosen to describe transformations of quantum states, and quantum variables and their substitutions were carefully treated. An operational semantics for qCCS based on exact (strong) bisimulation and an approximation version of bisimulation were presented for qCCS. Based on [13], several kind of bisimulations were presented for qCCS, such as probabilistic bisimulation[6], a kind of weak probabilistic bisimulation[15], open bisimulation [16] and symbolic bisimulation [14] [17]. These bisimulations provided qCCS with more semantic models. In some bisimulations, not only pure quantum data, but also classical data could be involved in qCCS.

Several years ago, we proposed an axiomatization of quantum processes called qACP [19], which is a quantum generalization of process algebra ACP [4]. This work uses some results of the previous works, especially qCCS, in the following ways. (1) qACP still uses the concept of a quantum process configuration which is widely used in quantum process algebra area $\langle p, \varrho \rangle$ [6] [9] [10] [7] [8] [13] [15] [6] [17], which is usually consisted of a process term p and quantum state information ϱ of all (public) quantum information variables. (2) Like qCCS, quantum system is treated as an open system, and quantum operations are chosen to describe transformations of quantum states, and behave as the atomic actions of a pure quantum process. Quantum measurements are treated as quantum operations, so probabilistic bisimulations are avoided. In qACP, quantum and classical computing are unified with the same equational logic and the same structured operational semantics based on process configuration. Unlike CCS [2] and CSP [3], ACP-like process algebras [4] are logic on actions, and data are hidden behind actions [1]. For qACP, it is the duty of atomic actions – quantum operations and atomic quantum communication actions, to obey the rules of quantum mechanics, such as the non-cloning theorem (it is also suitable for ACP-like probabilistic process algebra in this paper).

But, qACP relies on the assumption of an open quantum system and is not suitable for a closed quantum system. For a closed quantum system, the basic two quantum operations are unitary operator and quantum measurement, so the probability are unavoidable. Thanks for Andova's ACP-like probabilistic process algebra [5], we go long the solution of qACP for an open quantum system to unify quantum and classical computing for a closed quantum system by use of ACP-like probabilistic process algebra.

Fortunately, quantum and classical computing in closed systems are unified with Andova's ACP-like probabilistic process algebra, which has the same equational logic and the same structured operational semantics based on quantum configuration. There are several innovations in this paper, we enumerate them as follows. (1) With the assumption for a closed quantum system, we solve the modeling of the three main components in a closed quantum system: unitary operator, quantum measurement and quantum entanglement, with a full support for quantum and classical computing in closed systems. (2) We still use the framework of a quantum process configuration $\langle p, \varrho \rangle$, and let the quantum part ϱ be the outcomes of execution of p to examine and observe the function of the basic theory of quantum mechanics. We establish the relationship between probabilistic quantum bisimilarity and classical probabilistic bisimilarity, including strong probabilistic bisimilarity and weak probabilistic bisimilarity, which makes an axiomatization of quantum processes possible. With this framework, quantum and classical computing mixed processes are unified with the same structured operational semantics. (3) We establish a series of axiomatizations of quantum process algebra, including Fully Probabilistic Quantum Basic process algebra $fpqBPA$, Basic Quantum Process Algebra with Probabilistic Choice $pqBPA$, Probabilistic Basic Quantum Process Algebra with Projection $pqBPA + PR$, Probabilistic Quantum Variant of ACP $pqACP^+$, Fully Probabilistic Quantum Process Algebra with Abstraction $fpqBPA_\tau$, and some extensions, including an extension for quantum entanglement $pqACP_{QE}^+$, for renaming $pqACP_{RN}^+$, for priorities $pqACP_\Theta^+$. These process algebras support nearly all main computation properties. (4) In this paper, quantum and classical computing in closed quantum systems are unified with the same equational logic and the same structured operational semantics under the framework of ACP-like probabilistic process algebra. This unification means that our work can be used widely for verification for quantum and classical computing mixed systems, for example, most quantum communication protocols. (5) ACP-like axiomatization also inherits the advantages of ACP, for example, modularity makes it can be extended in an elegant way.

This paper is organized as follows. In section 2, we introduce some preliminaries, including basic quantum mechanics, and especially a brief introduction to Andova's ACP-like probabilistic process algebra. Respec-

tively, we model unitary operator, quantum measurement and quantum entanglement under the framework of probabilistic process algebra in section 3. Quantum and classical computing in closed systems are unified in section 4. The applications of verification for quantum and classical computing mixed systems are shown in section 5, including verifications for quantum teleportation protocol, BB84 protocol and E91 protocol. In section 6, we show the advantage of modularity of ACP-like process algebra, including extensions of renaming and priorities. Finally, in section 7, we conclude this paper.

2. Preliminaries

For convenience of the reader, we introduce some basic concepts about basic linear algebra, basic quantum mechanics (Please refer to [18] for details), and probabilistic process algebra (Please refer to [5] for more details).

2.1. Basic Quantum Mechanics

Definition 2.1 (Hilbert space). An isolated physical system is associated with a Hilbert space, which is called the state space of the system. A finite-dimensional Hilbert space is a complex vector space \mathcal{H} together with an inner product, which is a mapping $\langle \cdot | \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbf{C}$ satisfying: (1) $\langle \varphi | \varphi \rangle \geq 0$ with equality if and only if $|\varphi\rangle = 0$; (2) $\langle \varphi | \psi \rangle = \langle \psi | \varphi \rangle^*$; (3) $\langle \varphi | \lambda_1 \psi_1 + \lambda_2 \psi_2 \rangle = \lambda_1 \langle \varphi | \psi_1 \rangle + \lambda_2 \langle \varphi | \psi_2 \rangle$, where \mathbf{C} is the set of complex numbers, and λ^* denotes the conjugate of λ ($\lambda \in \mathbf{C}$).

Definition 2.2 (Orthonormal basis). For any vector $|\psi\rangle$ in \mathcal{H} , the length $\|\psi\| = \sqrt{\langle \psi | \psi \rangle}$. A vector $|\psi\rangle$ with $\|\psi\| = 1$ is called a unit vector in its state space. An orthonormal basis of a Hilbert space \mathcal{H} is a basis $\{|i\rangle\}$ with

$$\langle i | j \rangle = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.3 (Trace of a linear operator). The trace of a linear operator A on \mathcal{H} is defined as

$$tr(A) = \sum_i \langle i | A | i \rangle.$$

Definition 2.4. (Tensor products). The state space of a composite system is the tensor product of the state space of its components. Let \mathcal{H}_1 and \mathcal{H}_2 be two Hilbert spaces, then their tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ consists of linear vectors $|\psi_1 \psi_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, where $\psi_1 \in \mathcal{H}_1$ and $\psi_2 \in \mathcal{H}_2$.

For two linear operator A_1 on Hilbert space \mathcal{H}_1 , A_2 on Hilbert space \mathcal{H}_2 , $A_1 \otimes A_2$ is defined as

$$(A_1 \otimes A_2)|\psi_1 \psi_2\rangle = A_1|\psi_1\rangle \otimes A_2|\psi_2\rangle$$

where $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$.

Let $|\varphi\rangle = \sum_i \alpha_i |\varphi_{1i} \varphi_{2i}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ and $|\psi\rangle = \sum_j \beta_j |\psi_{1j} \psi_{2j}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Then the inner product of $|\varphi\rangle$ and $|\psi\rangle$ is defined as follows.

$$\langle \varphi | \psi \rangle = \sum_{i,j} \alpha_i^* \beta_j \langle \varphi_{1i} | \psi_{1j} \rangle \langle \varphi_{2i} | \psi_{2j} \rangle.$$

Definition 2.5 (Density operator). A mixed state of quantum system is represented by a density operator. A density operator in \mathcal{H} is a linear operator ϱ satisfying: (1) ϱ is positive, that is, $\langle \psi | \varrho | \psi \rangle \geq 0$ for all $|\psi\rangle$; (2) $tr(\varrho) = 1$. Let $\mathcal{D}(\mathcal{H})$ denote the set of all positive operators on \mathcal{H} .

Definition 2.6 (Unitary operator). The evolution of a closed quantum system is described by a unitary operator on its state space. A unitary operator is a linear operator U on a Hilbert space \mathcal{H} with $U^\dagger U = \mathcal{I}_{\mathcal{H}}$,

No.	Axiom
A5	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
PrAC1	$x \boxplus_{\pi} y = y \boxplus_{1-\pi} x$
PrAC2	$x \boxplus_{\pi} (y \boxplus_{\rho} z) = (x \boxplus_{\frac{\pi}{\pi+\rho-\pi\rho}} y) \boxplus_{\pi+\rho-\pi\rho} z$
PrAC3	$x \boxplus_{\pi} x = x$
PrAC4	$(x \boxplus_{\pi} y) \cdot z = x \cdot z \boxplus_{\pi} y \cdot z$

Table 1. Axioms for $fpBPA$

No.	Axiom
A1	$x + y = y + x$
A2	$(x + y) + z = x + (y + z)$
AA3	$a + a = a$
A4	$(x + y) \cdot z = x \cdot z + y \cdot z$
PrAC5	$(x \boxplus_{\pi} y) + z = (x + z) \boxplus_{\pi} (y + z)$

Table 2. Axioms for $pBPA_{-\delta}$

where $\mathcal{I}_{\mathcal{H}}$ is the identity operator on \mathcal{H} and U^{\dagger} is the adjoint of U . The unitary transformation U applied on the closed system ϱ has the effect $\varrho \rightarrow U\varrho U^{\dagger}$.

Definition 2.7 (Quantum measurement). Observation of quantum system is a quantum measurement represented by a Hermitian Operator M on the associated Hilbert space of the closed system. A quantum measurement consists of a collection of measurement operators $M = \sum_m m P_m$, where m is the measurement outcomes and P_m is the projector onto the eigenspace of M with eigenvalue m . The probability of getting m with an initial state $\varrho = |\psi\rangle$ and $p(m) = \text{tr}(P_m \varrho) = \langle \psi | P_m | \psi \rangle$, after measurement, the state of the system given the outcome m becomes $P_m \varrho P_m / \text{tr}(P_m \varrho) = P_m |\psi\rangle / \sqrt{p(m)}$.

2.2. Probabilistic Process Algebra

In the following, the variables x, x', y, y', z, z' range over the collection of process terms, s, s', t, t', u, u' are closed items, τ is the special constant silent step, δ is the special constant deadlock, A is the collection of atomic actions, atomic actions $a, b \in A$, $A_{\delta} = A \cup \{\delta\}$, $A_{\tau} = A \cup \{\tau\}$. And the predicate $\xrightarrow{a} \checkmark$ represents successful termination after execution of the action a .

Probabilistic process algebra [5] distinguishes probabilistic choice and non-deterministic choice to introduce a probabilistic version of ACP. In [5], there are several probabilistic process algebra working in a modular way (we do not concern the part of probabilistic process algebra with discrete time):

1. Fully Probabilistic Basic process algebra $fpBPA$: it consists of a (finite) set of constants, $A = \{a, b, c, \dots\}$ and two binary operators, sequential composition operator \cdot and probabilistic choice operator \boxplus_{π} ($\pi \in (0, 1)$). The axiom system is as Table 1 shows.
2. Basic Process Algebra with Probabilistic Choice (without deadlock) $pBPA_{-\delta}$: it consists of the signature of $fpBPA$ and the non-deterministic choice (alternative composition) operator $+$. The axiom system is shown in Table 2.
3. Basic Process Algebra with Probabilistic Choice $pBPA$: it is an extension of $pBPA_{-\delta}$ with the inaction process (deadlock process) by adding a new deadlock constant δ . The axiom system of $pBPA$ is shown in Table 3.
4. Probabilistic Basic Process Algebra with Projection $pBPA + PR$: it is an extension of $pBPA$ with the projection operator Π_n , which is used to approximate infinite processes, that is, the n -th projection of a process p is a process that behaves exactly like p till at most n steps are executed. The axiom system is shown in Table 4.

No.	Axiom
A6	$x + \delta = x$
A7	$\delta \cdot x = \delta$

Table 3. Axioms for $pBPA$

No.	Axiom
PR1	$\Pi_n(a) = a$
PR2	$\Pi_1(a \cdot x) = a$
PR3	$\Pi_{n+1}(a \cdot x) = a \cdot \Pi_n(x)$
PR4	$\Pi_n(x + y) = \Pi_n(x) + \Pi_n(y)$
prPR	$\Pi_n(x \boxplus_\rho y) = \Pi_n(x) \boxplus_\rho \Pi_n(y)$

Table 4. Axioms for $pBPA + PR$

5. Probabilistic Variant of ACP $pACP^+$: it is an extension of $pBPA$ by adding a variant of asynchronous probabilistic parallel composition \parallel . The asynchronous probabilistic parallel composition \parallel is defined by three auxiliary operators: merge with memory \llbracket , left merge \ll , and communication merge \mid . The axiom system is shown in Table 5.
6. Fully Probabilistic Process Algebra with Abstraction $fpBPA_\tau$: it is an extension of $fpBPA$ by adding the new silent constant τ and the abstraction operator τ_I for $I \subseteq A$, where A is the set of atomic action. The axiom system of $fpBPA_\tau$ is in Table 6. Note that the presence of both probabilistic choice and non-deterministic choice, and also abstraction at the same time is still missing, and its corresponding axiomatization remains an open problem.
7. To abstract away internal cycle (zero probability for infinite τ sequences), the following set of probabilistic verification rules PVR_n with $n \geq 1$ for fairness are defined. $fpBPA_\tau$ extended with these rules for fairness is called $fpBPA_\tau + PVR_1 + PVR_2 + \dots$.

No.	Axiom
PrMM1	$x \parallel y = (x, x) \llbracket (y, y)$
PrMM2	$(x \boxplus_\pi x', z) \llbracket (y, w) = (x, z) \llbracket (y, w) \boxplus_\pi (x', z) \llbracket (y, w)$
PrMM3	$(x, z) \llbracket (y \boxplus_\pi y', w) = (x, z) \llbracket (y, w) \boxplus_\pi (x, z) \llbracket (y', w)$
PrMM4	$x = x + x, y = y + y \Rightarrow (x, z) \llbracket (y, w) = x \ll w + y \ll z + x \mid y$
CF	$a \mid b = \gamma(a, b)$
CM2	$a \ll x = a \cdot x$
CM3	$a \cdot x \ll y = a \cdot (x \parallel y)$
CM4	$(x + y) \ll z = x \ll z + y \ll z$
PrCM1	$(x \boxplus_\pi y) \ll z = x \ll z \boxplus_\pi y \ll z$
CM5	$a \mid b \cdot x = (a \mid b) \cdot x$
CM6	$a \cdot x \mid b = (a \mid b) \cdot x$
CM7	$a \cdot x \mid b \cdot y = (a \mid b) \cdot (x \parallel y)$
PrCM2	$(x \boxplus_\pi y) \mid z = x \mid z \boxplus_\pi y \mid z$
PrCM3	$x \mid (y \boxplus_\pi z) = x \mid y \boxplus_\pi x \mid z$
D1	$\partial_H(a) = a$ if $a \notin H$
D2	$\partial_H(a) = \delta$ if $a \in H$
D3	$\partial_H(x + y) = \partial_H(x) + \partial_H(y)$
D4	$\partial_H(x \cdot y) = \partial_H(x) \cdot \partial_H(y)$
PrD5	$\partial_H(x \boxplus_\pi y) = \partial_H(x) \boxplus_\pi \partial_H(y)$
PrCM4	$z = z + z \Rightarrow (x + y) \mid z = x \mid z + y \mid z$
PrCM5	$z = z + z \Rightarrow z \mid (x + y) = z \mid x + z \mid y$

Table 5. Axioms for $pACP^+$

No.	Axiom
T1	$x \cdot \tau = x$
TI0	$\tau_I(\tau) = \tau$
TI1	$\tau_I(a) = a$ if $a \notin I$
TI2	$\tau_I(a) = \tau$ if $a \in I$
TI4	$\tau_I(x \cdot y) = \tau_I(x) \cdot \tau_I(y)$
PrTI	$\tau_I(x \boxplus_\pi y) = \tau_I(x) \boxplus_\pi \tau_I(y)$

Table 6. Axioms for $fpBPA_\tau$

$$\frac{X_1 = i \cdot X_1 \boxplus_{\pi_1} Y_1, \tau \neq i \in I}{\tau \cdot \tau_I(X_1) = \tau \cdot \tau_I(Y_1)} \quad (PVR_1)$$

$$X_1 = i_1 \cdot X_2 \boxplus_{\pi_1} Y_1$$

$$X_2 = i_2 \cdot X_3 \boxplus_{\pi_2} Y_2$$

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$$X_{n-1} = i_{n-1} \cdot X_n \boxplus_{\pi_{n-1}} Y_{n-1}$$

$$\frac{X_n = i_n \cdot X_1 \boxplus_{\pi_n} Y_n, \{\tau\} \neq \{i_1, i_2, \dots, i_n\} \subseteq I \cup \{\tau\}}{\tau \cdot \tau_I(X_1) = \tau \cdot (\tau_I(Y_1) \boxplus_{\alpha_1} \tau_I(Y_2) \boxplus_{\alpha_2} \dots \boxplus_{\alpha_{n-2}} \tau_I(Y_{n-1}) \boxplus_{\alpha_{n-1}} \tau_I(Y_n))} \quad (PVR_n)$$

where $\alpha_1 = \frac{1-\pi_1}{1-\pi_1 \cdot \pi_2 \cdot \dots \cdot \pi_n}$, $\alpha_j = \frac{\pi_1 \cdot \dots \cdot \pi_{j-1} (1-\pi_j)}{1-\pi_1 \cdot \pi_2 \cdot \dots \cdot \pi_n}$ for $1 \leq j \leq n$ and $\pi_k \in (0, 1)$ for $1 \leq k \leq n$.

$pACP^+$ is a theory for structured specification of probabilistic system. The operational semantics of $pACP^+$ is defined by a term deduction system, which consists of an extended set of constants (each atomic action a has a dynamic counterpart \check{a}) and its deduction rules include two kinds of transition type: probabilistic and action transition. Instead of labelling probabilistic transitions, the probability distribution function is defined, which gives a probability with which a probabilistic transition may occur. The concept of probabilistic bisimulation is used in the construction of the term models, and soundness and completeness of the term models with respect to the axiom system of $pACP^+$ is proven.

In the following, \rightsquigarrow denotes probabilistic transition, and action transition labelled by an atomic action $a \in A$, \xrightarrow{a} and $\xrightarrow{a} \checkmark$. $x \xrightarrow{a} p$ means that by performing action a process x evolves into p ; while $x \xrightarrow{a} \checkmark$ means that x performs an a action and then terminates. $p \rightsquigarrow x$ denotes that process p chooses to behave like process x with a non-zero probability $\pi > 0$.

Let PRA denotes the above probabilistic algebra generally. $\mathcal{SP}(PRA)$ denotes the set of all closed terms over the signature Σ_{PRA} . Terms in $\mathcal{D}(PRA)$ represent processes with a trivial probability distribution—only one process is assigned the non-zero probability 1.

The finite and infinite processes are denoted $\mathbb{PT}(PRA)$ and $\mathbb{PT}^{(\infty)}(PRA)$. Processes that make probabilistic transitions will be called static processes, denoted $\mathbb{SP}^{(\infty)}(PRA)$. The set of dynamic processes that make action transitions or deadlock, denoted $\mathbb{DP}^{(\infty)}(PRA)$. Processes that perform only one trivial probabilistic transition from processes with non-trivial probabilistic transitions is denoted $\mathbb{D}^{(\infty)}(PRA)$. The counter part of process p is denoted \check{p} which represents a dynamic processes, and $p \rightsquigarrow \check{p}$ with probability 1. That is, for every $p \in \mathbb{D}^{(\infty)}(PRA)$, there is a $\check{p} \in \mathbb{DP}^{(\infty)}(PRA)$ such that $p \rightsquigarrow \check{p}$ with probability 1. A probability distribution function (PDF) is a map $\mu : \mathbb{PT}^{(\infty)}(PRA) \times \mathbb{PT}^{(\infty)}(PRA) \rightarrow \langle 0, 1 \rangle$ and μ^* is the cumulative probability distribution function (cPDF). The deduction system \mathcal{T}_{PRA} contains $\check{\Sigma}_{PRA} = (A \cup \check{A} \cup A_{\{X|E\}}$, operators of PRA), and (probabilistic and action) deductive transition rules for PRA .

We retype the main definitions and conclusions as follows. The (probabilistic and action) transition rules of $pBPA$, $pBPA + PR$ and $pACP^+$ are not retyped, and also the definition of PDF μ of $pBPA$, $pBPA + PR$ and $pACP^+$ are not retyped, for details, please refer to [5].

Definition 2.8 (Basic terms). The set of basic terms of $pBPA$, $\mathcal{B}(pBPA)$, is inductively defined with the help of an intermediate set $\mathcal{B}_+(pBPA)$ [5]:

1. $A \cup \{\delta\} \subseteq \mathcal{B}_+(pBPA) \subset \mathcal{B}(pBPA)$;
2. $a \in A, t \in \mathcal{B}(pBPA) \Rightarrow a \cdot t \in \mathcal{B}_+(pBPA)$;
3. $t, s \in \mathcal{B}_+(pBPA) \Rightarrow t + s \in \mathcal{B}_+(pBPA)$;
4. $t, s \in \mathcal{B}(pBPA) \Rightarrow t \boxplus_\pi s \in \mathcal{B}_+(pBPA)$ for $\pi \in \langle 0, 1 \rangle$.

Theorem 2.9 (Elimination theorem of $pBPA$). Let p be a closed $pBPA$ term, then there is a closed basic $pBPA$ term q such that $pBPA \vdash p = q$.

Theorem 2.10 (Elimination theorem of $pACP^+$). Let p be a closed $pACP^+$ term, then there is a closed $pBPA$ term q such that $pACP^+ \vdash p = q$.

Definition 2.11 ((Strong) probabilistic bisimulation). Let R be an equivalence relation on $\mathbb{PT}^{(\infty)}(PRA)$. R is a probabilistic bisimulation if:

1. if pRq and $p \rightsquigarrow s$ then there is a term t such that $q \rightsquigarrow t$ and sRt ;
2. if sRt and $s \xrightarrow{a} p$ for some $a \in A$, then there is a term q such that $t \xrightarrow{a} q$ and pRq ;
3. if sRt and $s \xrightarrow{a} \surd$, then $t \xrightarrow{a} \surd$;
4. if pRq , then $\mu(p, M) = \mu(q, M)$ for each $M \in \mathbb{PT}^{(\infty)}(PRA)/R$.

If there is a probabilistic bisimulation R such that pRq , then p is probabilistically bisimilar to q , denoted by $p \leftrightarrow q$.

Theorem 2.12 (Soundness of $pBPA + PR$). Let x and y be $pBPA + PR$ terms. If $pBPA + PR \vdash x = y$ then $x \leftrightarrow y$.

Theorem 2.13 (AIP^- in $\mathbb{PT}^{(\infty)}(pBPA + PR)$). If for all $n \geq 1$, $\Pi_n(p) \leftrightarrow \Pi_n(q)$, then $p \leftrightarrow q$.

Theorem 2.14 (Soundness of $pBPA$). Let x and y be closed $pBPA$ terms. If $pBPA \vdash x = y$ then $x \leftrightarrow y$.

Theorem 2.15 (Completeness of $pBPA$). Let z and u be closed $pBPA$ terms, if $z \leftrightarrow u$, then $pBPA \vdash z = u$.

Theorem 2.16 (Soundness of $pACP^+$). Let p and q be closed $pACP^+$ terms. If $pACP^+ \vdash p = q$ then $p \leftrightarrow q$.

Theorem 2.17 (Completeness of $pACP^+$). Let z and u be closed $pACP^+$ terms, if $z \leftrightarrow u$, then $pACP^+ \vdash z = u$.

For the operational semantics of $fpBPA_\tau + PVR_1 + PVR_2 + \dots$, a probabilistic branching bisimulation is defined to construct the term models.

An action transition may have the termination state, denoted NIL as its incoming state. We also retype the main definitions and conclusions as follows.

Definition 2.18. A fully probabilistic graph g is a tuple $(S_p \cup S_n \cup \{NIL\}, \rightsquigarrow, \rightarrow, \mu, root)$ consisting of:

- a countable set S_p of probabilistic states,
- a countable set S_n of action states such that $S_p \cap S_n = \emptyset$ and $NIL \notin S_p \cup S_n$,
- $root \in S_p$,
- a relation $\rightsquigarrow \subseteq S_p \times S_n$,
- a function $\rightarrow: S_n \rightarrow (S_p \cup \{NIL\}) \times A_\tau$,
- a partial function $\mu: S_p \times S_n \rightarrow \langle 0, 1 \rangle$ such that $\mu(p, n)$ is defined iff $(p, n) \in \rightsquigarrow$ for $(p, n) \in S_p \times S_n$ and $\sum_{n \in S_n} \mu(p, n) = 1$ for any $p \in S_p$.

Definition 2.19 (Probabilistic branching bisimulation). Let g and h be fully probabilistic graphs. If R is an equivalence relation on $S_g \cup S_h \cup \{NUL_g, NIL_h\}$ such that:

1. $(root(g), root(h)) \in R$;
2. if $(p, q) \in R$ and $p \rightsquigarrow s$ then either
 - (a) $(s, q) \in R$ or

- (b) there are v, t such that $(p, v), (s, t) \in R$ and $q \xRightarrow{\tau^*} v \rightsquigarrow t$ or $q \xrightarrow{\tau} \cdot \xRightarrow{\tau^*} v \rightsquigarrow t$;
- 3. if $(p, q) \in R$ and $p \xrightarrow{a} s$ then either
 - (a) $a = \tau$ and $(s, q) \in R$ or
 - (b) there are v, t such that $(p, v), (s, t) \in R$ and $q \xRightarrow{\tau^*} \cdot \rightsquigarrow v \xrightarrow{a} t$ or $q(\xrightarrow{\tau} \cdot \rightsquigarrow)^* v \xrightarrow{a} t$;
- 4. there is an equivalence relation \tilde{R} on $Entry_R(g) \cup Entry_R(h)$ such that $\tilde{R} \subseteq R$ and
 - (a) $(root(g), root(h)) \in \tilde{R}$;
 - (b) if $(p, q) \in \tilde{R}$ then for any $C \in NextEntry_{C_{\tilde{R}}}(p) \cup NextEntry_{C_{\tilde{R}}}(q)$ and for any $a \in A$, $Prob_{[p]_R}(p, \tau^*, C) = Prob_{[q]_R}(q, \tau^*, C)$ and $Prob_{[p]_R}(p, \tau^* a, C) = Prob_{[q]_R}(q, \tau^* a, C)$;

where $Prob$ is the probability measure, $\xRightarrow{\tau^*}$ is the transitive and reflexive closure of \rightsquigarrow , $Entry_R(g)$ is the set of entries of the fully probabilistic graph g with a given equivalence relation R on g , $[p]_R$ is the equivalence class of the probabilistic state p in g . Then (R, \tilde{R}) is a probabilistic branching bisimulation relation between g and h . We write \xleftrightarrow{pb} if there is a probabilistic bisimulation R, \tilde{R} between g and h .

For additional conditions of probabilistic root branching bisimulation $g \xleftrightarrow{prb} h$, there is a probabilistic branching bisimulation (R, \tilde{R}) between g and h such that:

- 1. if $root(g) \rightsquigarrow p$ then there is q in h such that $root(h) \rightsquigarrow q$ and $(p, q) \in R$;
- 2. if $root(h) \rightsquigarrow q$ then there is p in g such that $root(g) \rightsquigarrow p$ and $(p, q) \in R$;
- 3. if $root(g) \rightsquigarrow p \xrightarrow{a} s$ for $a \in A_\tau$ then there are q and t in h such that $root(h) \rightsquigarrow q \xrightarrow{a} t$ and $(p, q) \in R$ and $(s, t) \in R$;
- 4. if $root(h) \rightsquigarrow q \xrightarrow{a} t$ for $a \in A_\tau$ then there are p and s in g such that $root(g) \rightsquigarrow p \xrightarrow{a} s$ and $(p, q) \in R$ and $(s, t) \in R$.

Theorem 2.20 (Soundness of $fpBPA_\tau + PVR_1 + PVR_2 + \dots$). G/\xleftrightarrow{prb} is a model of $fpBPA_\tau + PVR_1 + PVR_2 + \dots$.

3. Probabilistic Process Algebra for Closed Quantum Systems

Quantum operations (super operators) are used to describe the dynamics of an open quantum system. As done in quantum process algebras, qCCS [13] and qACP [19], quantum operations are deemed as the basic actions, so, a probabilistic algebra is avoided. But, in a closed quantum system, the basic action to describe the dynamics is unitary operator, and quantum measurement with a probabilistic nature is unavoidable (see basic quantum mechanics in section 2.1).

In this section, we discuss the main ingredients in a closed quantum system: unitary operator, quantum measurement, and quantum entanglement. We try to model them under the framework of probabilistic process algebra (see in section 2.2, and for details in [5]) and form a probabilistic quantum process algebra for pure quantum computing.

3.1. Unitary Operator in Probabilistic Process Algebra

In this pure probabilistic quantum process algebra, the basic atomic actions are all unitary operators, there is not classical actions involved (in section 4, we will discuss actions mixed by unitary operations and classical actions). In section 3.2 and section 3.3, we will discuss quantum measurement and quantum entanglement respectively, we will see that they do not affect the assumption about unitary operators as the basic atomic actions. And we use α, β to denote unitary operators in contrast to classical actions a, b , the variables v, w range over the set A of atomic unitary operators, $\alpha, \beta \in A$. And $A_\delta = A \cup \{\delta\}$, $A_\tau = A \cup \{\tau\}$.

The probabilistic quantum process algebra $PQRA$ is also defined by a term deduction system in a modular way, including $pqBPA$, $pqBPA + PR$, $pqACP^+$ and $fpqBPA_\tau$, which have the same operators and the same axiom systems shown in Table 1–6 as the corresponding PRA , including $pBPA$, $pBPA + PR$, $pACP^+$ and

$fpBPA_\tau$, but different atomic actions (quantum unitary operators in contrast to classical atomic actions). The meaning of atomic actions does not affect the properties of probabilistic process algebra ($PQRA$ and PRA). The corresponding properties of $pqBPA$, $pqBPA + PR$, $pqACP^+$ and $fpqBPA_\tau$ are same as those of $pBPA$, $pBPA + PR$, $pACP^+$ and $fpBPA_\tau$, such as elimination property (For details, please refer to [5]).

Indeed, The difference between $PQRA$ and PRA is their operational semantics, which is caused by the distinction of a unitary operator in quantum processes and a classical atomic action in classical computing. In quantum processes, to avoid the abuse of quantum information which may violate the no-cloning theorem, a quantum process configuration $\langle p, \varrho \rangle$ [6] [9] [10] [7] [8] [13] [15] [6] [17] [19] is usually consisted of a process term p and quantum state information ϱ of all (public) quantum information variables. Since ACP-like process algebras are based on actions, quantum information variables are not explicitly defined and hidden behind atomic actions (unitary operators and atomic quantum communication actions), more importantly, the state information ϱ is the effects of execution of a series of unitary operators on involved quantum systems, the execution of a series of atomic actions should not only obey the restrictions of the structure of the process terms, but also those of quantum mechanics principles, such as the no-cloning theorem. So, the operational semantics of quantum processes should be defined based on quantum process configuration $\langle p, \varrho \rangle$, in which $\varrho = \varsigma$ of two state information ϱ and ς means equality under the framework of quantum computing, that is, these two quantum processes are in the same quantum state.

So, for an atomic unitary operator α , without consideration of any probability, there should be an action transition rule as follows.

$$\overline{\langle \alpha, \varrho \rangle} \xrightarrow{\alpha} \overline{\langle \sqrt{}, \varrho' \rangle}$$

where $\varrho' = \alpha \varrho \alpha^\dagger$.

Firstly, we introduce some concepts and conclusions about operational semantics for probabilistic quantum processes.

Definition 3.1 (Quantum process configuration). A quantum process configuration is defined to be a pair $\langle p, \varrho \rangle$, where p is a (probabilistic) process (graph) called structural part of the configuration, and $\varrho \in \mathcal{D}(\mathcal{H})$ specifies the current state of the environment, which is called its quantum part.

Definition 3.2 ((Strong) probabilistic quantum bisimulation). Let R be an equivalence relation on $(\mathbb{PT}^{(\infty)}(PQRA), \varrho)$. R is a (strong) probabilistic quantum bisimulation if:

1. if $\langle p, \varrho \rangle R \langle q, \varrho \rangle$ and $\langle p, \varrho \rangle \rightsquigarrow \langle s, \varrho \rangle$ then there is a configuration $\langle t, \varrho \rangle$ such that $\langle q, \varrho \rangle \rightsquigarrow \langle t, \varrho \rangle$ and $\langle s, \varrho \rangle R \langle t, \varrho \rangle$;
2. if $\langle s, \varrho \rangle R \langle t, \varrho \rangle$ and $\langle s, \varrho \rangle \xrightarrow{\alpha} \langle p, \varrho' \rangle$ for some $\alpha \in A$, then there is a configuration $\langle q, \varrho \rangle$ such that $\langle t, \varrho \rangle \xrightarrow{\alpha} \langle q, \varrho' \rangle$ and $\langle p, \varrho' \rangle R \langle q, \varrho' \rangle$;
3. if $\langle s, \varrho \rangle R \langle t, \varrho \rangle$ and $\langle s, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{}, \varrho' \rangle$, then $\langle t, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{}, \varrho' \rangle$;
4. if $\langle p, \varrho \rangle R \langle q, \varrho \rangle$, then $\mu(p, M) = \mu(q, M)$ for each $M \in \mathbb{PT}^{(\infty)}(PQRA)/R$.

If there is a probabilistic quantum bisimulation R such that $\langle p, \varrho \rangle R \langle q, \varrho \rangle$, then $\langle p, \varrho \rangle$ is probabilistically quantum bisimilar to $\langle q, \varrho \rangle$, denoted by $\langle p, \varrho \rangle \stackrel{p}{\sim} \langle q, \varrho \rangle$.

In the following, we give the transition rules based on quantum configurations for $pqBPA + PR$, $pqBPA$, $pqACP^+$, including sequential composition operator \cdot , probabilistic choice operator \boxplus_π , alternative composition operator $+$, encapsulation operator ∂_H , projection Π_n , parallel composition \parallel , merge with memory \llbracket , left merge \ll and communication merge \mid .

The probabilistic transition rules for $pqBPA$ based on quantum configuration are as follows:

$$\begin{array}{c} \overline{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle} \quad \overline{\langle \delta, \varrho \rangle \rightsquigarrow \langle \check{\delta}, \varrho \rangle} \\[10pt] \frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle}{\langle x \cdot y, \varrho \rangle \rightsquigarrow \langle x' \cdot y, \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle, \langle y, \varrho \rangle \rightsquigarrow \langle y', \varrho \rangle}{\langle x + y, \varrho \rangle \rightsquigarrow \langle x' + y', \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \rightsquigarrow \langle z, \varrho \rangle}{\langle x \boxplus_\pi y, \varrho \rangle \rightsquigarrow \langle z, \varrho \rangle, \langle y \boxplus_{1-\pi} x, \varrho \rangle \rightsquigarrow \langle z, \varrho \rangle} \end{array}$$

The action transition rules for $pqBPA$ based on quantum configuration are as follows.

$$\begin{array}{c}
\frac{}{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle}{\langle x \cdot y, \varrho \rangle \xrightarrow{\alpha} \langle x' \cdot y, \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle x \cdot y, \varrho \rangle \xrightarrow{\alpha} \langle y, \varrho' \rangle} \\
\frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle}{\langle x + y, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle, \langle y + x, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle x + y, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle y + x, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}
\end{array}$$

The probabilistic transition rules for projection in $pqBPA + PR$ are as follows.

$$\frac{\langle \langle t_X | E \rangle, \varrho \rangle \rightsquigarrow \langle u, \varrho \rangle}{\langle \langle X | E \rangle, \varrho \rangle \rightsquigarrow \langle u, \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle}{\langle \Pi_n(x), \varrho \rangle \rightsquigarrow \langle \Pi_n(x'), \varrho \rangle}$$

The action transition rules for projection in $pqBPA + PR$ are as follows.

$$\frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle}{\langle \Pi_{n+1}(x), \varrho \rangle \xrightarrow{\alpha} \langle \Pi_n(x'), \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \Pi_n(x), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle}{\langle \Pi_1(x), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

The probabilistic transition rules for $pqACP^+$ are as follows.

$$\begin{array}{c}
\frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle, \langle y, \varrho \rangle \rightsquigarrow \langle y', \varrho \rangle}{\langle x \parallel y, \varrho \rangle \rightsquigarrow \langle x' \parallel y + y' \parallel x + x' \mid y', \varrho \rangle} \\
\frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle, \langle y, \varrho \rangle \rightsquigarrow \langle y', \varrho \rangle}{\langle \langle (x, z) \rangle [(y, w), \varrho] \rightsquigarrow \langle x' \parallel w + y' \parallel z + x' \mid y', \varrho \rangle} \\
\frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle}{\langle x \parallel y, \varrho \rangle \rightsquigarrow \langle x' \parallel y, \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle, \langle y, \varrho \rangle \rightsquigarrow \langle y', \varrho \rangle}{\langle x \mid y, \varrho \rangle \rightsquigarrow \langle x' \mid y', \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle}{\langle \partial_H(x), \varrho \rangle \rightsquigarrow \langle \partial_H(x'), \varrho \rangle}
\end{array}$$

In quantum computing, the communication of quantum data should be processed carefully to obey the quantum no-cloning theorem. We assume that all quantum variables are utilized by reference, the communication actions are all classical actions, but not unitary operators. We define a collection of communication actions denoted C , and let communication actions $a, b, c \in C$. The action transition rules for $pqACP^+$ are as follows.

$$\begin{array}{c}
\frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle}{\langle x \parallel y, \varrho \rangle \xrightarrow{\alpha} \langle x' \parallel y, \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{a} \langle x', \varrho \rangle}{\langle x \parallel y, \varrho \rangle \xrightarrow{a} \langle x' \parallel y, \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle x \parallel y, \varrho \rangle \xrightarrow{\alpha} \langle y, \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle}{\langle x \parallel y, \varrho \rangle \xrightarrow{a} \langle y, \varrho \rangle} \\
\frac{\langle x, \varrho \rangle \xrightarrow{a} \langle x', \varrho \rangle, \langle y, \varrho \rangle \xrightarrow{b} \langle y', \varrho \rangle, \gamma(a, b) = c}{\langle x \mid y, \varrho \rangle \xrightarrow{c} \langle x' \parallel y', \varrho \rangle} \\
\frac{\langle x, \varrho \rangle \xrightarrow{a} \langle x', \varrho \rangle, \langle y, \varrho \rangle \xrightarrow{b} \langle \surd, \varrho \rangle, \gamma(a, b) = c}{\langle x \mid y, \varrho \rangle \xrightarrow{c} \langle x', \varrho \rangle, \langle y \mid x, \varrho \rangle \xrightarrow{c} \langle x', \varrho \rangle} \\
\frac{\langle x, \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle, \langle y, \varrho \rangle \xrightarrow{b} \langle \surd, \varrho \rangle, \gamma(a, b) = c}{\langle x \mid y, \varrho \rangle \xrightarrow{c} \langle \surd, \varrho \rangle} \\
\frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle, \alpha \notin H}{\langle \partial_H(x), \varrho \rangle \xrightarrow{\alpha} \langle \partial_H(x'), \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{a} \langle x', \varrho \rangle, a \notin H}{\langle \partial_H(x), \varrho \rangle \xrightarrow{a} \langle \partial_H(x'), \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \alpha \notin H}{\langle \partial_H(x), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle, a \notin H}{\langle \partial_H(x), \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle}
\end{array}$$

The definitions of PDF μ and cPDF μ^* in $pqBPA$, $pqBPA + PR$ and $pqACP^+$ are the same as those in probabilistic process algebra $pBPA$, $pBPA + PR$ and $pACP^+$, and the conclusions about PDF μ and cPDF μ^* in $pBPA$, $pBPA + PR$ and $pACP^+$ still hold in $pqBPA$, $pqBPA + PR$ and $pqACP^+$. For details, please refer to [5], and we do not retype here.

Then we can get the first obvious but important conclusion as follows.

Proposition 3.3. \Leftrightarrow_q implies \Leftrightarrow with respect to $pqBPA$, $pqBPA + PR$ and $pqACP^+$.

Proof. From the definition of \Leftrightarrow_q (see in section 3.1) and the definition of \Leftrightarrow (see in section 2.2), it is obvious that \Leftrightarrow_q adds additional conditions about quantum information state ϱ into \Leftrightarrow , with respect to $pqBPA$, $pqBPA + PR$ and $pqACP^+$. So, let x and y be $pqBPA$, or $pqBPA + PR$, or $pqACP^+$ terms, $\langle x, \varrho \rangle \Leftrightarrow_q \langle y, \varrho \rangle \Rightarrow x \Leftrightarrow y$, as desired. \square

Theorem 3.4 (Congruence theorem of $pqBPA + PR$, $pqBPA$ and $pqACP^+$). \Leftrightarrow_q is a congruence relation on $\mathbb{PT}(pqBPA + PT)$, $\mathbb{PT}(pqBPA)$ and $\mathbb{PT}(pqACP^+)$ with respect to the $+$, \cdot , \boxplus_π , Π_n , $\llbracket \cdot \rrbracket$, $\llbracket \cdot \rrbracket$, $\llbracket \cdot \rrbracket$, and ∂_H operators.

Proof. It is obvious based on the following facts:

1. operators $+$, \cdot , \boxplus_π , Π_n , $\llbracket \cdot \rrbracket$, $\llbracket \cdot \rrbracket$, $\llbracket \cdot \rrbracket$, and ∂_H are all defined on $\mathbb{PT}(pqBPA + PT)$, $\mathbb{PT}(pqBPA)$ and $\mathbb{PT}(pqACP^+)$, which are the same as $\mathbb{PT}(pBPA + PT)$, $\mathbb{PT}(pBPA)$ and $\mathbb{PT}(pACP^+)$ respectively;
2. from Proposition 3.3, we know that \Leftrightarrow_q implies \Leftrightarrow with respect to $pqBPA$, $pqBPA + PR$ and $pqACP^+$.
3. \Leftrightarrow is a congruence relation on $\mathbb{PT}(pBPA + PT)$, $\mathbb{PT}(pBPA)$ and $\mathbb{PT}(pACP^+)$ with respect to the $+$, \cdot , \boxplus_π , Π_n , $\llbracket \cdot \rrbracket$, $\llbracket \cdot \rrbracket$, $\llbracket \cdot \rrbracket$, and ∂_H operators;
4. the quantum part ϱ in a quantum configuration $\langle p, \varrho \rangle$ satisfies the congruence relation with respect to the $+$, \cdot , \boxplus_π , Π_n , $\llbracket \cdot \rrbracket$, $\llbracket \cdot \rrbracket$, $\llbracket \cdot \rrbracket$, and ∂_H operators. It is trivial and we omit it.

\square

Theorem 3.5 (Soundness of $pqBPA + PR$). Let x and y be $pqBPA + PR$ terms. If $pqBPA + PR \vdash x = y$ then $\langle x, \varrho \rangle \Leftrightarrow_q \langle y, \varrho \rangle$.

Proof. It is already proven that \Leftrightarrow_q is an equivalent and congruent relation on $\mathbb{PT}^{(\infty)}(pBPA + PR)$ (see details in [5]), and $pBPA + PR \vdash x = y$ then $x \Leftrightarrow y$ (that is, the conditions on PDF μ are same in \Leftrightarrow_q and \Leftrightarrow , see in section 2.2 and [5] for details), we only need to prove that the quantum information ϱ related parts are also sound.

It is sufficient to examine every axiom in the axiom systems (in Table 1, 2, 3, 4) for $pqBPA + PR$ is sound.

- **Axiom A1.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle p+q, \varrho \rangle, \langle q+p, \varrho \rangle) = R_1$ with $p, q \in \mathbb{DP}^{(\infty)}(pqBPA + PR)$, and $Eq(\langle u+v, \varrho \rangle, \langle v+u, \varrho \rangle) = R_2$ with $u, v \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$. By use of the probabilistic transition rules and the action transition rules for alternative operator $+$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle u+v, \varrho \rangle \rightsquigarrow \langle u'+v', \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle v+u, \varrho \rangle \rightsquigarrow \langle v'+u', \varrho \rangle}$$

With the assumptions $u+v = v+u$ and $u'+v' = v'+u'$, we get R_2 satisfies condition 1 in the definition of \Leftrightarrow_q (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p+q, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle, \langle q+p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{\cdot}, \varrho' \rangle}{\langle p+q, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{\cdot}, \varrho' \rangle, \langle q+p, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{\cdot}, \varrho' \rangle}$$

With the assumption $p+q = q+p$, we get R_1 satisfies conditions 2 and 3 in the definition of \Leftrightarrow_q (see Definition 3.2).

So, $R = \Leftrightarrow_q$, as desired.

- **Axiom A2.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle (p+q) + s, \varrho \rangle, \langle p + (q+s), \varrho \rangle) = R_1$ with $p, q, s \in \mathbb{DP}^{(\infty)}(pqBPA + PR)$, and $Eq(\langle (u+v) + w, \varrho \rangle, \langle u + (v+w), \varrho \rangle) = R_2$ with $u, v, w \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$. By use of the probabilistic transition rules and the action transition rules for alternative operator $+$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle, \langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle}{\langle (u+v) + w, \varrho \rangle \rightsquigarrow \langle (u' + v') + w', \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle, \langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle}{\langle u + (v+w), \varrho \rangle \rightsquigarrow \langle u' + (v' + w'), \varrho \rangle}$$

With the assumptions $(u+v) + w = u + (v+w)$ and $(u' + v') + w' = u' + (v' + w')$, we get R_2 satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle (p+q) + s, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p + (q+s), \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle (p+q) + s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle p + (q+s), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $(p+q) + s = p + (q+s)$, we get R_1 satisfies conditions 2 and 3 in the definition of \xleftrightarrow{q} (see Definition 3.2).

So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom AA3.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \alpha + \alpha, \varrho \rangle, \langle \alpha, \varrho \rangle) = R_1$, and $Eq(\langle \check{\alpha} + \check{\alpha}, \varrho \rangle, \langle \check{\alpha}, \varrho \rangle) = R_2$. By use of the probabilistic transition rules and the action transition rules for atomic unitary operator and alternative operator $+$, we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle, \langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha + \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha} + \check{\alpha}, \varrho \rangle}$$

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}$$

With the assumptions $\alpha + \alpha = \alpha$ and $\check{\alpha} + \check{\alpha} = \check{\alpha}$, we get R_1 satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2).

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} + \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $\check{\alpha} + \check{\alpha} = \check{\alpha}$, we get R_2 satisfies conditions 2 and 3 in the definition of \xleftrightarrow{q} (see Definition 3.2).

So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom A4.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle (p+q) \cdot s, \varrho \rangle, \langle p \cdot s + q \cdot s, \varrho \rangle) = R_1$ with

$p, q \in \mathbb{DP}^{(\infty)}(pqBPA + PR)$, $s \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$, and $Eq(\langle (u+v) \cdot s, \varrho \rangle, \langle u \cdot s + v \cdot s, \varrho \rangle) = R_2$ with $u, v, s \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules and the action transition rules for sequential composition \cdot , alternative operator $+$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle (u+v) \cdot s, \varrho \rangle \rightsquigarrow \langle (u' + v') \cdot s, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle u \cdot s + v \cdot s, \varrho \rangle \rightsquigarrow \langle u' \cdot s + v' \cdot s, \varrho \rangle}$$

With the assumptions $(u+v) \cdot s = u \cdot s + v \cdot s$ and $(u' + v') \cdot s = u' \cdot s + v' \cdot s$, we get R_2 satisfies condition 1 in the definition of \xrightarrow{q} (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle (p+q) \cdot s, \varrho \rangle \xrightarrow{\alpha} \langle p' \cdot s, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p \cdot s + q \cdot s, \varrho \rangle \xrightarrow{\alpha} \langle p' \cdot s, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle (p+q) \cdot s, \varrho \rangle \xrightarrow{\alpha} \langle s, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle p \cdot s + q \cdot s, \varrho \rangle \xrightarrow{\alpha} \langle s, \varrho' \rangle}$$

With the assumptions $(p+q) \cdot s = p \cdot s + q \cdot s$, we get R_1 satisfies conditions 2 and 3 in the definition of \xrightarrow{q} (see Definition 3.2).

So, $R = \xrightarrow{q}$, as desired.

- **Axiom A5.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle (p \cdot q) \cdot s, \varrho \rangle, \langle p \cdot (q \cdot s), \varrho \rangle) = R_1$ with $p \in \mathbb{DP}^{(\infty)}(pqBPA + PR)$, $q, s \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$, and $Eq(\langle (u \cdot q) \cdot s, \varrho \rangle, \langle u \cdot (q \cdot s), \varrho \rangle) = R_2$ with $u, q, s \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules and the action transition rules for sequential composition \cdot , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle (u \cdot q) \cdot s, \varrho \rangle \rightsquigarrow \langle (u' \cdot q) \cdot s, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \cdot (q \cdot s), \varrho \rangle \rightsquigarrow \langle u' \cdot (q \cdot s), \varrho \rangle}$$

With the assumptions $(u \cdot q) \cdot s = u \cdot (q \cdot s)$ and $(u' \cdot q) \cdot s = u' \cdot (q \cdot s)$, we get R_2 satisfies condition 1 in the definition of \xrightarrow{q} (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle (p \cdot q) \cdot s, \varrho \rangle \xrightarrow{\alpha} \langle (p' \cdot q) \cdot s, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p \cdot (q \cdot s), \varrho \rangle \xrightarrow{\alpha} \langle p' \cdot (q \cdot s), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle (p \cdot q) \cdot s, \varrho \rangle \xrightarrow{\alpha} \langle q \cdot s, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle p \cdot (q \cdot s), \varrho \rangle \xrightarrow{\alpha} \langle q \cdot s, \varrho' \rangle}$$

With the assumptions $(p \cdot q) \cdot s = p \cdot (q \cdot s)$ and $(p' \cdot q) \cdot s = p' \cdot (q \cdot s)$, we get R_1 satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom A6.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle p + \check{\delta}, \varrho \rangle, \langle p, \varrho \rangle) = R_1$ with $p \in \mathbb{DP}^{(\infty)}(pqBPA + PR)$, and $Eq(\langle u + \delta, \varrho \rangle, \langle u, \varrho \rangle) = R_2$ with $u \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules and the action transition rules for alternative operator $+$ and the constant δ , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle \delta, \varrho \rangle \rightsquigarrow \langle \check{\delta}, \varrho \rangle}{\langle u + \delta, \varrho \rangle \rightsquigarrow \langle u' + \check{\delta}, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}$$

With the assumptions $u + \delta = u$ and $u' + \check{\delta} = u'$, we get R_2 satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p + \check{\delta}, \varrho \rangle \xrightarrow{\alpha} \langle p' + \check{\delta}, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle p + \check{\delta}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumptions $p + \check{\delta} = p$ and $p' + \check{\delta} = p'$, we get R_1 satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom A7.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \delta \cdot p, \varrho \rangle, \langle \delta, \varrho \rangle) = R_1$ with $p \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$, and $Eq(\langle \check{\delta} \cdot u, \varrho \rangle, \langle \check{\delta}, \varrho \rangle) = R_2$ with $u \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules and the action transition rules for sequential operator \cdot and the constant δ , we get:

$$\frac{\langle \delta, \varrho \rangle \rightsquigarrow \langle \check{\delta}, \varrho \rangle}{\langle \delta \cdot p, \varrho \rangle \rightsquigarrow \langle \check{\delta} \cdot p, \varrho \rangle}$$

$$\frac{\langle \delta, \varrho \rangle \rightsquigarrow \langle \check{\delta}, \varrho \rangle}{\langle \delta, \varrho \rangle \rightsquigarrow \langle \check{\delta}, \varrho \rangle}$$

With the assumptions $\delta \cdot p = \delta$ and $\check{\delta} \cdot p = \check{\delta}$, we get R_1 satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

There are not action transition rules for the constant $\check{\delta}$, which mean that $\check{\delta}$ leads to inaction processes, accompany with the action transition rules of sequential composition \cdot , $\check{\delta} \cdot p \nrightarrow$ and $\check{\delta} \nrightarrow$ all lead to inaction

processes, with the assumption $\check{\delta} \cdot p = \check{\delta}$, we get R_2 satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom PrAC1.** For a relation R , suppose that $Eq(\langle u \boxplus_{\pi} v, \varrho \rangle, \langle v \boxplus_{1-\pi} u, \varrho \rangle) = R$ with $u, v \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_{π} , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \boxplus_{\pi} v, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v \boxplus_{1-\pi} u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}$$

With the assumption $u \boxplus_{\pi} v = v \boxplus_{1-\pi} u$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2). Note that, for probabilistic choice operator \boxplus_{π} , there are not action transition rules.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom PrAC2.** For a relation R , suppose that $Eq(\langle u \boxplus_{\pi} (v \boxplus_{\rho} w), \varrho \rangle, \langle (u \boxplus_{\frac{\pi}{\pi+\rho-\pi\rho}} v) \boxplus_{\pi+\rho-\pi\rho} w, \varrho \rangle) = R$ with $u, v, w \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_{π} , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \boxplus_{\pi} (v \boxplus_{\rho} w), \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle (u \boxplus_{\frac{\pi}{\pi+\rho-\pi\rho}} v) \boxplus_{\pi+\rho-\pi\rho} w, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}$$

With the assumption $u \boxplus_{\pi} (v \boxplus_{\rho} w) = (u \boxplus_{\frac{\pi}{\pi+\rho-\pi\rho}} v) \boxplus_{\pi+\rho-\pi\rho} w$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2). Note that, for probabilistic choice operator \boxplus_{π} , there are not action transition rules.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom PrAC3.** For a relation R , suppose that $Eq(\langle u \boxplus_{\pi} u, \varrho \rangle, \langle u, \varrho \rangle) = R$ with $u \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_{π} , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \boxplus_{\pi} u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}$$

With the assumption $u \boxplus_{\pi} u = u$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2). Note that, for probabilistic choice operator \boxplus_{π} , there are not action transition rules.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom PrAC4.** For a relation R , suppose that $Eq(\langle (u \boxplus_{\pi} v) \cdot w, \varrho \rangle, \langle u \cdot w \boxplus_{\pi} v \cdot w, \varrho \rangle) = R$ with $u, v, w \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_{π} and sequential operator \cdot , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle (u \boxplus_{\pi} v) \cdot w, \varrho \rangle \rightsquigarrow \langle u' \cdot w, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \cdot w \boxplus_{\pi} v \cdot w, \varrho \rangle \rightsquigarrow \langle u' \cdot w, \varrho \rangle}$$

With the assumption $(u \boxplus_{\pi} v) \cdot w = u \cdot w \boxplus_{\pi} v \cdot w$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2). Note that, for probabilistic choice operator \boxplus_{π} , there are not action transition rules.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom PrAC5.** For a relation R , suppose that $Eq(\langle (u \boxplus_{\pi} v) + w, \varrho \rangle, \langle u + w \boxplus_{\pi} v + w, \varrho \rangle) = R$ with $u, v, w \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_π and alternative operator $+$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle (u \boxplus_\pi v) + w, \varrho \rangle \rightsquigarrow \langle u' + w, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u + w \boxplus_\pi v + w, \varrho \rangle \rightsquigarrow \langle u' + w, \varrho \rangle}$$

With the assumption $(u \boxplus_\pi v) + w = u + w \boxplus_\pi v + w$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2). Note that, for probabilistic choice operator \boxplus_π , there are not action transition rules. So, $R = \leftrightarrow_q$, as desired.

- **Axiom PR1.** For a relation $R_n (n \geq 1) = Eq(\langle \Pi_n(\alpha), \varrho \rangle, \langle \alpha, \varrho \rangle) \cup Eq(\langle \Pi_n(\check{\alpha}), \varrho \rangle, \langle \check{\alpha}, \varrho \rangle)$. By use of the probabilistic transition rules and the action transition rules for atomic unitary operator and projection operator $\Pi_n (n \geq 1)$, we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \Pi_n(\alpha), \varrho \rangle \rightsquigarrow \langle \Pi_n(\check{\alpha}), \varrho \rangle}$$

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}$$

With the assumptions $\Pi_n(\alpha) = \alpha$ and $\Pi_n(\check{\alpha}) = \check{\alpha}$, we get R_n satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \Pi_n(\check{\alpha}), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $\Pi_n(\check{\alpha}) = \check{\alpha}$, we get R_n satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R_n = \leftrightarrow_q$, as desired.

- **Axiom PR2.** For a relation $R_1 = Eq(\langle \Pi_1(\alpha \cdot u), \varrho \rangle, \langle \alpha, \varrho \rangle) \cup Eq(\langle \Pi_1(\check{\alpha} \cdot u), \varrho \rangle, \langle \check{\alpha}, \varrho \rangle)$, with $u \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$. By use of the probabilistic transition rules and the action transition rules for sequential operator \cdot and projection operator $\Pi_n (n \geq 1)$, we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \Pi_1(\alpha \cdot u), \varrho \rangle \rightsquigarrow \langle \Pi_1(\check{\alpha} \cdot u), \varrho \rangle}$$

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}$$

With the assumptions $\Pi_1(\alpha \cdot u) = \alpha$ and $\Pi_1(\check{\alpha} \cdot u) = \check{\alpha}$, we get R_1 satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \Pi_1(\check{\alpha} \cdot u), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $\Pi_1(\check{\alpha} \cdot u) = \check{\alpha}$, we get R_1 satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R_1 = \leftrightarrow_q$, as desired.

- **Axiom PR3.** For a relation $R_n (n \geq 1) = Eq(\langle \Pi_{n+1}(\alpha \cdot u), \varrho \rangle, \langle \alpha \cdot \Pi_n(u), \varrho \rangle) \cup Eq(\langle \Pi_{n+1}(\check{\alpha} \cdot u), \varrho \rangle, \langle \check{\alpha} \cdot \Pi_n(u), \varrho \rangle)$, with $u \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules and the action transition rules for atomic unitary operator, sequential operator \cdot and projection operator $\Pi_n (n \geq 1)$, we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \Pi_{n+1}(\alpha \cdot u), \varrho \rangle \rightsquigarrow \langle \Pi_{n+1}(\check{\alpha} \cdot u), \varrho \rangle}$$

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \cdot \Pi_n(u), \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot \Pi_n(u), \varrho \rangle}$$

With the assumptions $\Pi_{n+1}(\alpha \cdot u) = \alpha \cdot \Pi_n(u)$ and $\Pi_{n+1}(\check{\alpha} \cdot u) = \check{\alpha} \cdot \Pi_n(u)$, we get R_n satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \Pi_{n+1}(\check{\alpha} \cdot u), \varrho \rangle \xrightarrow{\alpha} \langle \Pi_n(u), \varrho' \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot \Pi_n(u), \varrho \rangle \xrightarrow{\alpha} \langle \Pi_n(u), \varrho' \rangle}$$

With the assumption $\Pi_{n+1}(\check{\alpha} \cdot u) = \check{\alpha} \cdot \Pi_n(u)$, we get R_n satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R_n = \leftrightarrow_q$, as desired.

- **Axiom PR4.** For a relation $R_n (n \geq 1) = Eq(\langle \Pi_n(u+v), \varrho \rangle, \langle \Pi_n(u) + \Pi_n(v), \varrho \rangle) \cup Eq(\langle \Pi_n(p+q), \varrho \rangle, \langle \Pi_n(p) + \Pi_n(q), \varrho \rangle)$, with $u, v \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$ and $p, q \in \mathbb{DP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules and the action transition rules for alternative operator $+$ and projection operator $\Pi_n (n \geq 1)$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle \Pi_n(u+v), \varrho \rangle \rightsquigarrow \langle \Pi_n(u') + \Pi_n(v'), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle \Pi_n(u) + \Pi_n(v), \varrho \rangle \rightsquigarrow \langle \Pi_n(u') + \Pi_n(v'), \varrho \rangle}$$

With the assumptions $\Pi_n(u+v) = \Pi_n(u) + \Pi_n(v)$ and $\Pi_n(u' + v') = \Pi_n(u') + \Pi_n(v')$, we get R_n satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \Pi_n(p+q), \varrho \rangle \xrightarrow{\alpha} \langle \Pi_{n-1}(p'), \varrho' \rangle} \quad n \geq 2$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \Pi_n(p) + \Pi_n(q), \varrho \rangle \xrightarrow{\alpha} \langle \Pi_{n-1}(p'), \varrho' \rangle} \quad n \geq 2$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \Pi_n(p+q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \quad n \geq 2$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \Pi_n(p) + \Pi_n(q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \quad n \geq 2$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \Pi_1(p+q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \Pi_1(p) + \Pi_1(q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $\Pi_n(p+q) = \Pi_n(p) + \Pi_n(q)$, we get R_n satisfies conditions 2 and 3 in the definition of \xleftrightarrow{q} (see Definition 3.2).

So, $R_n = \xleftrightarrow{q}$, as desired.

- **Axiom prPR.** For a relation $R_n (n \geq 1) = Eq(\langle \Pi_n(u \boxplus_\pi v), \varrho \rangle, \langle \Pi_n(u) \boxplus_\pi \Pi_n(v), \varrho \rangle)$, with $u, v \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_π and projection operator $\Pi_n (n \geq 1)$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \Pi_n(u \boxplus_\pi v), \varrho \rangle \rightsquigarrow \langle \Pi_n(u'), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \Pi_n(u) \boxplus_\pi \Pi_n(v), \varrho \rangle \rightsquigarrow \langle \Pi_n(u'), \varrho \rangle}$$

With the assumption $\Pi_n(u \boxplus_\pi v) = \Pi_n(u) \boxplus_\pi \Pi_n(v)$, we get R_n satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2).

Note that, there are not action transition rules for probabilistic choice operator \boxplus_π .

So, $R_n = \xleftrightarrow{q}$, as desired.

□

Theorem 3.6 (AIP^- in $\mathbb{PT}^{(\infty)}(pqBPA + PR)$). If for all $n \geq 1$, $\langle \Pi_n(p), \varrho \rangle \xleftrightarrow{q} \langle \Pi_n(q), \varrho \rangle$, then $\langle p, \varrho \rangle \xleftrightarrow{q} \langle q, \varrho \rangle$.

Proof. The conditions on PDF μ are same in \xleftrightarrow{q} and $\xleftrightarrow{\cdot}$ (see in section 2.2 and [5] for details), we only need to process the quantum information ϱ related parts on projections.

The proof is similar to that of Theorem (AIP^- in $\mathbb{PT}^{(\infty)}(pBPA + PR)$), please refer to [5] for details. Next, we only give the skeleton of the proof.

For a relation $R = R_1 \cup R_2$, $R_1 = Eq(\langle u, \varrho \rangle, \langle v, \varrho \rangle)$, with $\langle \Pi_n(u), \varrho \rangle \xleftrightarrow{q} \langle \Pi_n(v), \varrho \rangle (n \geq 1)$ and $u, v \in \mathbb{SP}^{(\infty)}(pqBPA + PR)$, $R_2 = Eq(\langle p, \varrho \rangle, \langle q, \varrho \rangle)$, with $\langle \Pi_n(p), \varrho \rangle \xleftrightarrow{q} \langle \Pi_n(q), \varrho \rangle (n \geq 1)$ and $p, q \in \mathbb{DP}^{(\infty)}(pqBPA + PR)$.

For probabilistic transitions, surely, u and v have head normal forms (HNFs) (see in [5]). Suppose $\langle u, \varrho \rangle \rightsquigarrow \langle p, \varrho \rangle$, $p \equiv \check{u}_i$ for some i , and will eventually lead to $\langle v, \varrho \rangle \rightsquigarrow \langle q, \varrho \rangle$ for some q , and $\langle \Pi_n(p), \varrho \rangle \xleftrightarrow{q} \langle \Pi_n(q), \varrho \rangle$, $(\langle p, \varrho \rangle, \langle q, \varrho \rangle) \in R$.

For action transitions, p and q have head normal forms (HNFs) (see in [5]). Suppose $\langle p, \varrho \rangle \xrightarrow{\alpha} \langle u, \varrho' \rangle$ for some u , $u \equiv \check{u}_j$ for some j , and will eventually lead to $\langle q, \varrho \rangle \xrightarrow{\alpha} \langle v, \varrho' \rangle$ for some v , and $\langle \Pi_n(u), \varrho \rangle \xleftrightarrow{q} \langle \Pi_n(v), \varrho \rangle$, $(\langle u, \varrho' \rangle, \langle v, \varrho' \rangle) \in R$.

For action termination, $\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle$, then $\langle \Pi_1(p), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle$. Since $\langle \Pi_1(p), \varrho \rangle \xleftrightarrow{q} \langle \Pi_1(q), \varrho \rangle$, so, we get $\langle \Pi_1(q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle$. So, $\langle q, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle$. □

Theorem 3.7 (Soundness of pqBPA). Let x and y be closed $pqBPA$ terms. If $pqBPA \vdash x = y$ then $\langle x, \varrho \rangle \xleftrightarrow{q} \langle y, \varrho \rangle$.

Proof. By Theorem 3.5. □

Theorem 3.8 (Completeness of pqBPA). Let z and u are closed $pqBPA$ terms, if $\langle z, \varrho \rangle \xleftrightarrow{q} \langle u, \varrho \rangle$, then $pqBPA \vdash z = u$.

Proof. It is based on the following three facts: let z and u are closed $pqBPA$ terms,

1. $\langle z, \varrho \rangle \xleftrightarrow{q} \langle u, \varrho \rangle \Rightarrow z \xleftrightarrow{q} u$ by Proposition 3.3;
2. $z \xleftrightarrow{q} u \Rightarrow pBPA \vdash z = u$ (see in section 2.2, and for details in [5]);
3. the term systems of $pBPA$ and $pqBPA$ are same, including the same operators and the same axiom systems, but different atomic action meanings (semantics).

So, we get $\langle z, \varrho \rangle \xleftrightarrow{q} \langle u, \varrho \rangle \Rightarrow pqBPA \vdash z = u$, as desired. \square

Theorem 3.9 (Soundness of $pqACP^+$). Let p and q be closed $pqACP^+$ terms. If $pqACP^+ \vdash p = q$ then $\langle p, \varrho \rangle \xleftrightarrow{q} \langle q, \varrho \rangle$.

Proof. It is already proven that \xleftrightarrow{q} is an equivalent and congruent relation on $\mathbb{PT}^{(\infty)}(pACP^+)$ (see details in [5]), and $pACP^+ \vdash x = y$ then $x \xleftrightarrow{q} y$ (that is, the conditions on PDF μ are same in \xleftrightarrow{q} and \xleftrightarrow{q} , see in section 2.2 and [5] for details), we only need to prove that the quantum information ϱ related parts are also sound.

It is sufficient to examine every axiom in the axiom systems which are added to $pqBPA$ to obtain $pqACP^+$ (in Table 5) is sound.

- **Axiom CF.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle a \mid b, \varrho \rangle, \langle c, \varrho \rangle) = R_1$ with $c = \gamma(a, b)$, and $Eq(\langle \check{a} \mid \check{b}, \varrho \rangle, \langle \check{c}, \varrho \rangle) = R_2$ with $\check{c} = \gamma(\check{a}, \check{b})$.
By use of the probabilistic transition rules and the action transition rules for atomic communication action and communication merge \mid , we get:

$$\frac{\langle a, \varrho \rangle \rightsquigarrow \langle \check{a}, \varrho \rangle, \langle b, \varrho \rangle \rightsquigarrow \langle \check{b}, \varrho \rangle}{\langle a \mid b, \varrho \rangle \rightsquigarrow \langle \check{a} \mid \check{b}, \varrho \rangle}$$

$$\frac{\langle \check{a}, \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle, \langle \check{b}, \varrho \rangle \xrightarrow{b} \langle \surd, \varrho \rangle}{\langle \check{a} \mid \check{b}, \varrho \rangle \xrightarrow{c} \langle \surd, \varrho \rangle}$$

With the assumptions $a \mid b = c$ and $\check{a} \mid \check{b} = \check{c}$, we get R satisfies conditions 1, 2 and 3 in the definition of \xleftrightarrow{q} (see Definition 3.2).

So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom CM2.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle a \parallel u, \varrho \rangle, \langle a \cdot u, \varrho \rangle) = R_1$ with $u \in \mathbb{SP}(pqACP^+)$, and $Eq(\langle \check{a} \parallel u, \varrho \rangle, \langle \check{a} \cdot u, \varrho \rangle) = R_2$ with $u \in \mathbb{SP}(pqACP^+)$.
By use of the probabilistic transition rules and the action transition rules for atomic communication action a and left merge \parallel , we get:

$$\frac{\langle a, \varrho \rangle \rightsquigarrow \langle \check{a}, \varrho \rangle}{\langle a \parallel u, \varrho \rangle \rightsquigarrow \langle \check{a} \parallel u, \varrho \rangle}$$

$$\frac{\langle a, \varrho \rangle \rightsquigarrow \langle \check{a}, \varrho \rangle}{\langle a \cdot u, \varrho \rangle \rightsquigarrow \langle \check{a} \cdot u, \varrho \rangle}$$

$$\frac{\langle \check{a}, \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle}{\langle \check{a} \parallel u, \varrho \rangle \xrightarrow{a} \langle u, \varrho \rangle}$$

$$\frac{\langle \check{a}, \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle}{\langle \check{a} \cdot u, \varrho \rangle \xrightarrow{a} \langle u, \varrho \rangle}$$

With the assumptions $a \parallel u = a \cdot u$ and $\check{a} \parallel u = \check{a} \cdot u$, we get R satisfies conditions 1, 2 and 3 in the definition of \xleftrightarrow{q} (see Definition 3.2).

So, $R = \xleftrightarrow{q}$, as desired.

For a relation $R' = R'_1 \cup R'_2$, suppose that $Eq(\langle \alpha \parallel u, \varrho \rangle, \langle \alpha \cdot u, \varrho \rangle) = R'_1$ with $u \in \mathbb{SP}(pqACP^+)$, and $Eq(\langle \check{\alpha} \parallel u, \varrho \rangle, \langle \check{\alpha} \cdot u, \varrho \rangle) = R'_2$ with $u \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules and the action transition rules for atomic unitary operator α and left merge \parallel , we get:

$$\begin{array}{c}
\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \parallel u, \varrho \rangle \rightsquigarrow \langle \check{\alpha} \parallel u, \varrho \rangle} \\
\\
\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \cdot u, \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot u, \varrho \rangle} \\
\\
\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \parallel u, \varrho \rangle \xrightarrow{\alpha} \langle u, \varrho' \rangle} \\
\\
\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot u, \varrho \rangle \xrightarrow{\alpha} \langle u, \varrho' \rangle}
\end{array}$$

With the assumptions $\alpha \parallel u = \alpha \cdot u$ and $\check{\alpha} \parallel u = \check{\alpha} \cdot u$, we get R' satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R' = \leftrightarrow_q$, as desired.

- **Axiom CM3.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle a \cdot u \parallel v, \varrho \rangle, \langle a \cdot (u \parallel v), \varrho \rangle) = R_1$ with $u, v \in \mathbb{SP}(pqACP^+)$, and $Eq(\langle \check{a} \cdot u \parallel v, \varrho \rangle, \langle \check{a} \cdot (u \parallel v), \varrho \rangle) = R_2$ with $u, v \in \mathbb{SP}(pqACP^+)$. By use of the probabilistic transition rules and the action transition rules for atomic communication action a , sequential composition \cdot and left merge \parallel , we get:

$$\begin{array}{c}
\frac{\langle a, \varrho \rangle \rightsquigarrow \langle \check{a}, \varrho \rangle}{\langle a \cdot u \parallel v, \varrho \rangle \rightsquigarrow \langle \check{a} \cdot u \parallel v, \varrho \rangle} \\
\\
\frac{\langle a, \varrho \rangle \rightsquigarrow \langle \check{a}, \varrho \rangle}{\langle a \cdot (u \parallel v), \varrho \rangle \rightsquigarrow \langle \check{a} \cdot (u \parallel v), \varrho \rangle} \\
\\
\frac{\langle \check{a}, \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle}{\langle \check{a} \cdot u \parallel v, \varrho \rangle \xrightarrow{a} \langle u \parallel v, \varrho \rangle} \\
\\
\frac{\langle \check{a}, \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle}{\langle \check{a} \cdot (u \parallel v), \varrho \rangle \xrightarrow{a} \langle u \parallel v, \varrho \rangle}
\end{array}$$

With the assumptions $a \cdot u \parallel v = a \cdot (u \parallel v)$ and $\check{a} \cdot u \parallel v = \check{a} \cdot (u \parallel v)$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

For a relation $R' = R'_1 \cup R'_2$, suppose that $Eq(\langle \alpha \cdot u \parallel v, \varrho \rangle, \langle \alpha \cdot (u \parallel v), \varrho \rangle) = R'_1$ with $u, v \in \mathbb{SP}(pqACP^+)$, and $Eq(\langle \check{\alpha} \cdot u \parallel v, \varrho \rangle, \langle \check{\alpha} \cdot (u \parallel v), \varrho \rangle) = R'_2$ with $u, v \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules and the action transition rules for atomic unitary operator α , sequential composition \cdot and left merge \parallel , we get:

$$\begin{array}{c}
\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \cdot u \parallel v, \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot u \parallel v, \varrho \rangle} \\
\\
\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \cdot (u \parallel v), \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot (u \parallel v), \varrho \rangle} \\
\\
\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot u \parallel v, \varrho \rangle \xrightarrow{\alpha} \langle u \parallel v, \varrho' \rangle} \\
\\
\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot (u \parallel v), \varrho \rangle \xrightarrow{\alpha} \langle u \parallel v, \varrho' \rangle}
\end{array}$$

With the assumptions $\alpha \cdot u \parallel v = \alpha \cdot (u \parallel v)$ and $\check{\alpha} \cdot u \parallel v = \check{\alpha} \cdot (u \parallel v)$, we get R' satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R' = \leftrightarrow_q$, as desired.

- **Axiom CM4.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle (p+q) \parallel s, \varrho \rangle, \langle p \parallel s+q \parallel s, \varrho \rangle) = R_1$ with $p, q \in \mathbb{DP}(pqACP^+)$, $s \in \mathbb{SP}(pqACP^+)$, and $Eq(\langle (u+v) \parallel s, \varrho \rangle, \langle u \parallel s+v \parallel s, \varrho \rangle) = R_2$ with $u, v, s \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules and the action transition rules for left merge \parallel , alternative operator $+$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle (u+v) \parallel s, \varrho \rangle \rightsquigarrow \langle (u'+v') \parallel s, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle u \parallel s+v \parallel s, \varrho \rangle \rightsquigarrow \langle u' \parallel s+v' \parallel s, \varrho \rangle}$$

With the assumptions $(u+v) \parallel s = u \parallel s+v \parallel s$ and $(u'+v') \parallel s = u' \parallel s+v' \parallel s$, we get R_2 satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle (p+q) \parallel s, \varrho \rangle \xrightarrow{\alpha} \langle p' \parallel s, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p \parallel s+q \parallel s, \varrho \rangle \xrightarrow{\alpha} \langle p' \parallel s, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle (p+q) \parallel s, \varrho \rangle \xrightarrow{\alpha} \langle s, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle p \parallel s+q \parallel s, \varrho \rangle \xrightarrow{\alpha} \langle s, \varrho' \rangle}$$

With the assumptions $(p+q) \parallel s = p \parallel s+q \parallel s$, we get R_1 satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom PrCM1.** For a relation R , suppose that $Eq(\langle (u \boxplus_{\pi} v) \parallel w, \varrho \rangle, \langle u \parallel w \boxplus_{\pi} v \parallel w, \varrho \rangle) = R$ with $u, v, w \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_{π} and left merge \parallel , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle (u \boxplus_{\pi} v) \parallel w, \varrho \rangle \rightsquigarrow \langle u' \parallel w, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \parallel w \boxplus_{\pi} v \parallel w, \varrho \rangle \rightsquigarrow \langle u' \parallel w, \varrho \rangle}$$

With the assumption $(u \boxplus_{\pi} v) \parallel w = u \parallel w \boxplus_{\pi} v \parallel w$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2). Note that, for probabilistic choice operator \boxplus_{π} , there are not action transition rules.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom CM5.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle a \cdot u \mid b, \varrho \rangle, \langle c \cdot u, \varrho \rangle) = R_1$ with $c = \gamma(a, b)$ and $u \in \mathbb{SP}(pqACP^+)$, and $Eq(\langle \check{a} \cdot u \mid \check{b}, \varrho \rangle, \langle \check{c} \cdot u, \varrho \rangle) = R_2$ with $\check{c} = \gamma(\check{a}, \check{b})$ and $u \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules and the action transition rules for atomic communication action a, b, c , sequential composition \cdot and communication merge \mid , we get:

$$\begin{array}{c}
\frac{\langle a, \varrho \rangle \rightsquigarrow \langle \check{a}, \varrho \rangle, \langle b, \varrho \rangle \rightsquigarrow \langle \check{b}, \varrho \rangle}{\langle a \cdot u \mid b, \varrho \rangle \rightsquigarrow \langle \check{a} \cdot u \mid \check{b}, \varrho \rangle} \\
\\
\frac{\langle c, \varrho \rangle \rightsquigarrow \langle \check{c}, \varrho \rangle}{\langle c \cdot u, \varrho \rangle \rightsquigarrow \langle \check{c} \cdot u, \varrho \rangle} \\
\\
\frac{\langle \check{a}, \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle, \langle \check{b}, \varrho \rangle \xrightarrow{b} \langle \surd, \varrho \rangle}{\langle \check{a} \cdot u \mid \check{b}, \varrho \rangle \xrightarrow{c} \langle u, \varrho \rangle} \\
\\
\frac{\langle \check{c}, \varrho \rangle \xrightarrow{c} \langle \surd, \varrho \rangle}{\langle \check{c} \cdot u, \varrho \rangle \xrightarrow{c} \langle u, \varrho \rangle}
\end{array}$$

With the assumptions $a \cdot u \mid b = c \cdot u$ and $\check{a} \cdot u \mid \check{b} = \check{c} \cdot u$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom CM6.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle a \mid b \cdot u, \varrho \rangle, \langle c \cdot u, \varrho \rangle) = R_1$ with $c = \gamma(a, b)$ and $u \in \mathbb{SP}(pqACP^+)$, and $Eq(\langle \check{a} \mid \check{b} \cdot u, \varrho \rangle, \langle \check{c} \cdot u, \varrho \rangle) = R_2$ with $\check{c} = \gamma(\check{a}, \check{b})$ and $u \in \mathbb{SP}(pqACP^+)$. By use of the probabilistic transition rules and the action transition rules for atomic communication action a, b, c , sequential composition \cdot and communication merge \mid , we get:

$$\begin{array}{c}
\frac{\langle a, \varrho \rangle \rightsquigarrow \langle \check{a}, \varrho \rangle, \langle b, \varrho \rangle \rightsquigarrow \langle \check{b}, \varrho \rangle}{\langle a \mid b \cdot u, \varrho \rangle \rightsquigarrow \langle \check{a} \mid \check{b} \cdot u, \varrho \rangle} \\
\\
\frac{\langle c, \varrho \rangle \rightsquigarrow \langle \check{c}, \varrho \rangle}{\langle c \cdot u, \varrho \rangle \rightsquigarrow \langle \check{c} \cdot u, \varrho \rangle} \\
\\
\frac{\langle \check{a}, \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle, \langle \check{b}, \varrho \rangle \xrightarrow{b} \langle \surd, \varrho \rangle}{\langle \check{a} \mid \check{b} \cdot u, \varrho \rangle \xrightarrow{c} \langle u, \varrho \rangle} \\
\\
\frac{\langle \check{c}, \varrho \rangle \xrightarrow{c} \langle \surd, \varrho \rangle}{\langle \check{c} \cdot u, \varrho \rangle \xrightarrow{c} \langle u, \varrho \rangle}
\end{array}$$

With the assumptions $a \mid b \cdot u = c \cdot u$ and $\check{a} \mid \check{b} \cdot u = \check{c} \cdot u$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom CM7.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle a \cdot u \mid b \cdot v, \varrho \rangle, \langle c \cdot (u \parallel v), \varrho \rangle) = R_1$ with $c = \gamma(a, b)$ and $u, v \in \mathbb{SP}(pqACP^+)$, and $Eq(\langle \check{a} \cdot u \mid \check{b} \cdot v, \varrho \rangle, \langle \check{c} \cdot (u \parallel v), \varrho \rangle) = R_2$ with $\check{c} = \gamma(\check{a}, \check{b})$ and $u, v \in \mathbb{SP}(pqACP^+)$. By use of the probabilistic transition rules and the action transition rules for atomic communication action a, b, c , sequential composition \cdot and communication merge \parallel , we get:

$$\begin{array}{c}
\frac{\langle a, \varrho \rangle \rightsquigarrow \langle \check{a}, \varrho \rangle, \langle b, \varrho \rangle \rightsquigarrow \langle \check{b}, \varrho \rangle}{\langle a \cdot u \mid b \cdot v, \varrho \rangle \rightsquigarrow \langle \check{a} \cdot u \mid \check{b} \cdot v, \varrho \rangle} \\
\\
\frac{\langle c, \varrho \rangle \rightsquigarrow \langle \check{c}, \varrho \rangle}{\langle c \cdot (u \parallel v), \varrho \rangle \rightsquigarrow \langle \check{c} \cdot (u \parallel v), \varrho \rangle} \\
\\
\frac{\langle \check{a}, \varrho \rangle \xrightarrow{a} \langle \surd, \varrho \rangle, \langle \check{b}, \varrho \rangle \xrightarrow{b} \langle \surd, \varrho \rangle}{\langle \check{a} \cdot u \mid \check{b} \cdot v, \varrho \rangle \xrightarrow{c} \langle (u \parallel v), \varrho \rangle}
\end{array}$$

$$\frac{\langle \check{c}, \varrho \rangle \xrightarrow{c} \langle \check{\vee}, \varrho \rangle}{\langle \check{c} \cdot (u \parallel v), \varrho \rangle \xrightarrow{c} \langle (u \parallel v), \varrho \rangle}$$

With the assumptions $a \cdot u \mid b \cdot v = c \cdot (u \parallel v)$ and $\check{a} \cdot u \mid \check{b} \cdot v = \check{c} \cdot (u \parallel v)$, we get R satisfies conditions 1, 2 and 3 in the definition of \xleftrightarrow{q} (see Definition 3.2).

So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom PrCM2.** For a relation R , suppose that $Eq(\langle (u \boxplus_{\pi} v) \mid w, \varrho \rangle, \langle u \mid w \boxplus_{\pi} v \mid w, \varrho \rangle) = R$ with $u, v, w \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_{π} and communication merge \mid , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle (u \boxplus_{\pi} v) \mid w, \varrho \rangle \rightsquigarrow \langle u' \mid w, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \mid w \boxplus_{\pi} v \mid w, \varrho \rangle \rightsquigarrow \langle u' \mid w, \varrho \rangle}$$

With the assumption $(u \boxplus_{\pi} v) \mid w = u \mid w \boxplus_{\pi} v \mid w$, we get R satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2). Note that, for probabilistic choice operator \boxplus_{π} , there are not action transition rules. So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom PrCM3.** For a relation R , suppose that $Eq(\langle u \mid (v \boxplus_{\pi} w), \varrho \rangle, \langle u \mid v \boxplus_{\pi} u \mid w, \varrho \rangle) = R$ with $u, v, w \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_{π} and communication merge \mid , we get:

$$\frac{\langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle u \mid (v \boxplus_{\pi} w), \varrho \rangle \rightsquigarrow \langle u \mid v', \varrho \rangle}$$

$$\frac{\langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle u \mid v \boxplus_{\pi} u \mid w, \varrho \rangle \rightsquigarrow \langle u \mid v', \varrho \rangle}$$

With the assumption $u \mid (v \boxplus_{\pi} w) = u \mid v \boxplus_{\pi} u \mid w$, we get R satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2). Note that, for probabilistic choice operator \boxplus_{π} , there are not action transition rules. So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom PrMM1.** For a relation R , suppose that $Eq(\langle u \parallel v, \varrho \rangle, \langle (u, u) \parallel (v, v), \varrho \rangle) = R$ with $u, v \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules for operators \parallel and \llbracket , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle u \parallel v, \varrho \rangle \rightsquigarrow \langle u' \parallel v + v' \parallel u + u' \mid v', \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle (u, u) \parallel (v, v), \varrho \rangle \rightsquigarrow \langle u' \parallel v + v' \parallel u + u' \mid v', \varrho \rangle}$$

With the assumption $u \parallel v = (u, u) \parallel (v, v)$, we get R satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2). Note that, for operator \parallel and \llbracket , there are not action transition rules.

So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom PrMM2.** For a relation R , suppose that $Eq(\langle (u_1 \boxplus_{\pi} u_2, z) \parallel (v, w), \varrho \rangle, \langle (u_1, z) \parallel (v, w) \boxplus_{\pi} (u_2, z) \parallel (v, w), \varrho \rangle) = R$ with $u_1, u_2, v, z, w \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_{π} , and \llbracket , we get:

$$\frac{\langle u_1, \varrho \rangle \rightsquigarrow \langle u'_1, \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle (u_1 \boxplus_{\pi} u_2, z) \parallel (v, w), \varrho \rangle \rightsquigarrow \langle u'_1 \parallel w + v' \parallel z + u'_1 \mid v', \varrho \rangle}$$

$$\frac{\langle u_1, \varrho \rangle \rightsquigarrow \langle u'_1, \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle (u_1, z) \rangle [(v, w) \boxplus_\pi (u_2, z)] [(v, w), \varrho] \rightsquigarrow \langle u'_1 \parallel w + v' \parallel z + u'_1 \mid v', \varrho \rangle}$$

With the assumption $(u_1 \boxplus_\pi u_2, z) [(v, w) = (u_1, z)] [(v, w) \boxplus_\pi (u_2, z)] [(v, w), \varrho]$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2). Note that, for operator \boxplus_π and \llbracket , there are not action transition rules.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom PrMM3.** For a relation R , suppose that $Eq(\langle (u, z) \rangle [(v_1 \boxplus_\pi v_2, w), \varrho], \langle (u, z) \rangle [(v_1, w) \boxplus_\pi (u, z)] [(v_1, w), \varrho]) = R$ with $u_1, u_2, v, z, w \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_π , and \llbracket , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v_1, \varrho \rangle \rightsquigarrow \langle v'_1, \varrho \rangle}{\langle (u, z) \rangle [(v_1 \boxplus_\pi v_2, w), \varrho] \rightsquigarrow \langle u' \parallel w + v'_1 \parallel z + u' \mid v'_1, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v_1, \varrho \rangle \rightsquigarrow \langle v'_1, \varrho \rangle}{\langle (u, z) \rangle [(v_1, w) \boxplus_\pi (u, z)] [(v_1, w), \varrho] \rightsquigarrow \langle u' \parallel w + v'_1 \parallel z + u' \mid v'_1, \varrho \rangle}$$

With the assumption $(u, z) [(v_1 \boxplus_\pi v_2, w) = (u, z)] [(v_1, w) \boxplus_\pi (u, z)] [(v_1, w), \varrho]$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2). Note that, for operator \boxplus_π and \llbracket , there are not action transition rules.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom D1.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \partial_H(\alpha), \varrho \rangle, \langle \alpha, \varrho \rangle) = R_1$ with $\alpha \notin H$, and $Eq(\langle \partial_H(\check{\alpha}), \varrho \rangle, \langle \check{\alpha}, \varrho \rangle) = R_2$ with $\check{\alpha} \notin H$.

By use of the probabilistic transition rules and the action transition rules for atomic unitary operator and encapsulation operator ∂_H , we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \partial_H(\alpha), \varrho \rangle \rightsquigarrow \langle \partial_H(\check{\alpha}), \varrho \rangle}$$

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}$$

With the assumptions $\partial_H(\alpha) = \alpha$ and $\partial_H(\check{\alpha}) = \check{\alpha}$ with $\alpha, \check{\alpha} \notin H$, we get R_1 satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \partial_H(\check{\alpha}), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $\partial_H(\check{\alpha}) = \check{\alpha}$, we get R_2 satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom D2.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \partial_H(\alpha), \varrho \rangle, \langle \delta, \varrho \rangle) = R_1$ with $\alpha \in H$, and $Eq(\langle \partial_H(\check{\delta}), \varrho \rangle, \langle \check{\delta}, \varrho \rangle) = R_2$ with $\check{\delta} \in H$.

By use of the probabilistic transition rules and the action transition rules for atomic unitary operator and encapsulation operator ∂_H , we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\delta}, \varrho \rangle}{\langle \partial_H(\alpha), \varrho \rangle \rightsquigarrow \langle \partial_H(\check{\delta}), \varrho \rangle}$$

There are not action transition rules for the constant $\check{\delta}$, which mean that $\check{\delta}$ leads to inaction processes, accompany with the action transition rules of operator ∂_H , $\partial_H(\check{\delta}) \nrightarrow$ and $\check{\delta} \nrightarrow$ all lead to inaction processes.

With the assumptions $\partial_H(\alpha) = \delta$ and $\partial_H(\check{\alpha}) = \check{\delta}$ with $\alpha, \check{\alpha} \in H$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom D3.** For a relation $R = Eq(\langle \partial_H(u + v), \varrho \rangle, \langle \partial_H(u) + \partial_H(v), \varrho \rangle) \cup Eq(\langle \partial_H(p + q), \varrho \rangle, \langle \partial_H(p) + \partial_H(q), \varrho \rangle)$, with $u, v \in \mathbb{SP}(pqACP^+)$ and $p, q \in \mathbb{DP}(pqACP^+)$.
By use of the probabilistic transition rules and the action transition rules for alternative operator $+$ and encapsulation operator ∂_H , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle \partial_H(u + v), \varrho \rangle \rightsquigarrow \langle \partial_H(u') + \partial_H(v'), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle \partial_H(u) + \partial_H(v), \varrho \rangle \rightsquigarrow \langle \partial_H(u') + \partial_H(v'), \varrho \rangle}$$

With the assumptions $\partial_H(u + v) = \partial_H(u) + \partial_H(v)$ and $\partial_H(u' + v') = \partial_H(u') + \partial_H(v')$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \partial_H(p + q), \varrho \rangle \xrightarrow{\alpha} \langle \partial_H(p'), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \partial_H(p) + \partial_H(q), \varrho \rangle \xrightarrow{\alpha} \langle \partial_H(p'), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \partial_H(p + q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \partial_H(p) + \partial_H(q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $\partial_H(p + q) = \partial_H(p) + \partial_H(q)$, we get R satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

For the case of the atomic communication action a instead of atomic unitary operator α , the proof is similar to those of axioms *CM2* and *CM3* and is omitted.

- **Axiom D4.** For a relation $R = Eq(\langle \partial_H(u \cdot v), \varrho \rangle, \langle \partial_H(u) \cdot \partial_H(v), \varrho \rangle) \cup Eq(\langle \partial_H(p \cdot v), \varrho \rangle, \langle \partial_H(p) \cdot \partial_H(v), \varrho \rangle)$, with $u, v \in \mathbb{SP}(pqACP^+)$ and $p \in \mathbb{DP}(pqACP^+)$.
By use of the probabilistic transition rules and the action transition rules for sequential composition \cdot and encapsulation operator ∂_H , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \partial_H(u \cdot v), \varrho \rangle \rightsquigarrow \langle \partial_H(u') \cdot \partial_H(v), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \partial_H(u) \cdot \partial_H(v), \varrho \rangle \rightsquigarrow \langle \partial_H(u') \cdot \partial_H(v), \varrho \rangle}$$

With the assumptions $\partial_H(u \cdot v) = \partial_H(u) \cdot \partial_H(v)$ and $\partial_H(u' \cdot v) = \partial_H(u') \cdot \partial_H(v)$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \partial_H(p \cdot v), \varrho \rangle \xrightarrow{\alpha} \langle \partial_H(p') \cdot \partial_H(v), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \partial_H(p) \cdot \partial_H(v), \varrho \rangle \xrightarrow{\alpha} \langle \partial_H(p') \cdot \partial_H(v), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{\cdot}, \varrho' \rangle}{\langle \partial_H(p \cdot v), \varrho \rangle \xrightarrow{\alpha} \langle \partial_H(v), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{\cdot}, \varrho' \rangle}{\langle \partial_H(p) \cdot \partial_H(v), \varrho \rangle \xrightarrow{\alpha} \langle \partial_H(v), \varrho' \rangle}$$

With the assumption $\partial_H(p \cdot v) = \partial_H(p) \cdot \partial_H(v)$ and $\partial_H(p' \cdot v) = \partial_H(p') \cdot \partial_H(v)$, we get R satisfies conditions 2 and 3 in the definition of \xleftrightarrow{q} (see Definition 3.2).

So, $R = \xleftrightarrow{q}$, as desired.

For the case of the atomic communication action a instead of atomic unitary operator α , the proof is similar to those of axioms *CM2* and *CM3* and is omitted.

- **Axiom *PrD5*.** For a relation $R = Eq(\langle \partial_H(u \boxplus_{\pi} v), \varrho \rangle, \langle \partial_H(u) \boxplus_{\pi} \partial_H(v), \varrho \rangle)$, with $u, v \in \mathbb{SP}(pqACP^+)$. By use of the probabilistic transition rules for probabilistic choice operator \boxplus_{π} and encapsulation operator ∂_H , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \partial_H(u \boxplus_{\pi} v), \varrho \rangle \rightsquigarrow \langle \partial_H(u'), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \partial_H(u) \boxplus_{\pi} \partial_H(v), \varrho \rangle \rightsquigarrow \langle \partial_H(u'), \varrho \rangle}$$

With the assumption $\partial_H(u \boxplus_{\pi} v) = \partial_H(u) \boxplus_{\pi} \partial_H(v)$, we get R satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2).

Note that, there are not action transition rules for probabilistic choice operator \boxplus_{π} .

So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom *PrMM4*.** For a relation R , suppose that $Eq(\langle (u, z) \rangle[(v, w), \varrho], \langle u \parallel w + v \parallel z + u \mid v, \varrho \rangle) = R$ with $u = u + u$, $v = v + v$, and $u, v, z, w \in \mathbb{SP}(pqACP^+)$. By use of the probabilistic transition rules for operators \parallel , \mid and $\rangle[$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle (u, z) \rangle[(v, w), \varrho] \rightsquigarrow \langle u' \parallel w + v' \parallel z + u' \mid v', \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle u \parallel w + v \parallel z + u \mid v, \varrho \rangle \rightsquigarrow \langle u' \parallel w + v' \parallel z + u' \mid v', \varrho \rangle}$$

In the above two probabilistic transition rules, the condition $u = u + u$ ensures that $\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle$, $\langle u, \varrho \rangle \rightsquigarrow \langle u'', \varrho \rangle$, $\langle u', \varrho \rangle \xleftrightarrow{q} \langle u'', \varrho \rangle$. $v = v + v$ ensures the similar things.

With the assumption $\langle (u, z) \rangle[(v, w), \varrho] = \langle u \parallel w + v \parallel z + u \mid v, \varrho \rangle$, we get R satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2). Note that, for operator $\rangle[$, there are not action transition rules.

So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom *PrCM4*.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle (p + q) \mid s, \varrho \rangle, \langle p \mid s + q \mid s, \varrho \rangle) = R_1$ with $p, q, s \in \mathbb{DP}(pqACP^+)$, and $Eq(\langle (u + v) \mid w, \varrho \rangle, \langle u \mid w + v \mid w, \varrho \rangle) = R_2$ with $w = w + w$ and $u, v, w \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules and the action transition rules for communication merge \mid , alternative operator $+$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle, \langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle}{\langle (u + v) \mid w, \varrho \rangle \rightsquigarrow \langle (u' + v') \mid w', \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle, \langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle}{\langle u \mid w + v \mid w, \varrho \rangle \rightsquigarrow \langle u' \mid w' + v' \mid w', \varrho \rangle}$$

In the above two probabilistic transition rules, the condition $w = w + w$ ensures that $\langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle$, $\langle w, \varrho \rangle \rightsquigarrow \langle w'', \varrho \rangle$, $\langle w', \varrho \rangle \xleftrightarrow{q} \langle w'', \varrho \rangle$.

With the assumptions $(u + v) \mid w = u \mid w + v \mid w$ and $(u' + v') \mid w' = u' \mid w' + v' \mid w'$, we get R_2 satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2).

$$\begin{array}{c}
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle p', \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle s', \varrho \rangle}{\langle (p + q) \mid s, \varrho \rangle \xrightarrow{\gamma(a,b)} \langle p' \parallel s', \varrho \rangle} \\
\\
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle p', \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle s', \varrho \rangle}{\langle p \mid s + q \mid s, \varrho \rangle \xrightarrow{\gamma(a,b)} \langle p' \parallel s', \varrho \rangle} \\
\\
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle p', \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle \sqrt{\cdot}, \varrho \rangle}{\langle (p + q) \mid s, \varrho \rangle \xrightarrow{\gamma(a,b)} \langle p', \varrho \rangle} \\
\\
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle p', \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle \sqrt{\cdot}, \varrho \rangle}{\langle p \mid s + q \mid s, \varrho \rangle \xrightarrow{\gamma(a,b)} \langle p', \varrho \rangle} \\
\\
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle \sqrt{\cdot}, \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle \sqrt{\cdot}, \varrho \rangle}{\langle (p + q) \mid s, \varrho \rangle \xrightarrow{\gamma(a,b)} \langle \sqrt{\cdot}, \varrho \rangle} \\
\\
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle \sqrt{\cdot}, \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle \sqrt{\cdot}, \varrho \rangle}{\langle p \mid s + q \mid s, \varrho \rangle \xrightarrow{\gamma(a,b)} \langle \sqrt{\cdot}, \varrho \rangle}
\end{array}$$

With the assumptions $(p + q) \mid s = p \mid s + q \mid s$, we get R_1 satisfies conditions 2 and 3 in the definition of \xleftrightarrow{q} (see Definition 3.2).

So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom *PrCM5*.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle s \mid (p + q), \varrho \rangle, \langle s \mid p + s \mid q, \varrho \rangle) = R_1$ with $p, q, s \in \mathbb{DP}(pqACP^+)$, and $Eq(\langle w \mid (u + v), \varrho \rangle, \langle w \mid u + w \mid v, \varrho \rangle) = R_2$ with $w = w + w$ and $u, v, w \in \mathbb{SP}(pqACP^+)$.

By use of the probabilistic transition rules and the action transition rules for communication merge \mid , alternative operator $+$, we get:

$$\begin{array}{c}
\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle, \langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle}{\langle w \mid (u + v), \varrho \rangle \rightsquigarrow \langle w' \mid (u' + v'), \varrho \rangle} \\
\\
\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle, \langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle}{\langle w \mid u + w \mid v, \varrho \rangle \rightsquigarrow \langle w' \mid u' + w' \mid v', \varrho \rangle}
\end{array}$$

In the above two probabilistic transition rules, the condition $w = w + w$ ensures that $\langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle$, $\langle w, \varrho \rangle \rightsquigarrow \langle w'', \varrho \rangle$, $\langle w', \varrho \rangle \xleftrightarrow{q} \langle w'', \varrho \rangle$.

With the assumptions $w \mid (u + v) = w \mid u + w \mid v$ and $w' \mid (u' + v') = w' \mid u' + w' \mid v'$, we get R_2 satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2).

$$\begin{array}{c}
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle p', \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle s', \varrho \rangle}{\langle s \mid (p + q), \varrho \rangle \xrightarrow{\gamma(a,b)} \langle s' \parallel p', \varrho \rangle} \\
\\
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle p', \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle s', \varrho \rangle}{\langle s \mid p + s \mid q, \varrho \rangle \xrightarrow{\gamma(a,b)} \langle s' \parallel p', \varrho \rangle}
\end{array}$$

$$\begin{array}{c}
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle p', \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle \sqrt{\cdot}, \varrho \rangle}{\langle s \mid (p+q), \varrho \rangle \xrightarrow{\gamma(a,b)} \langle p', \varrho \rangle} \\
\\
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle p', \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle \sqrt{\cdot}, \varrho \rangle}{\langle s \mid p+s \mid q, \varrho \rangle \xrightarrow{\gamma(a,b)} \langle p', \varrho \rangle} \\
\\
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle \sqrt{\cdot}, \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle \sqrt{\cdot}, \varrho \rangle}{\langle s \mid (p+q), \varrho \rangle \xrightarrow{\gamma(a,b)} \langle \sqrt{\cdot}, \varrho \rangle} \\
\\
\frac{\langle p, \varrho \rangle \xrightarrow{a} \langle \sqrt{\cdot}, \varrho \rangle, \langle s, \varrho \rangle \xrightarrow{b} \langle \sqrt{\cdot}, \varrho \rangle}{\langle s \mid p+s \mid q, \varrho \rangle \xrightarrow{\gamma(a,b)} \langle \sqrt{\cdot}, \varrho \rangle}
\end{array}$$

With the assumptions $s \mid (p+q) = s \mid p+s \mid q$, we get R_1 satisfies conditions 2 and 3 in the definition of $\xleftrightarrow{a,b}_q$ (see Definition 3.2).

So, $R = \xleftrightarrow{a,b}_q$, as desired.

□

Theorem 3.10 (Completeness of $pqACP^+$). Let z and u are closed $pqACP^+$ terms, if $\langle z, \varrho \rangle \xleftrightarrow{a,b}_q \langle u, \varrho \rangle$, then $pqACP^+ \vdash z = u$.

Proof. It is based on the following three facts: let z and u are closed $pqACP^+$ terms,

1. $\langle z, \varrho \rangle \xleftrightarrow{a,b}_q \langle u, \varrho \rangle \Rightarrow z \xleftrightarrow{a,b} u$ by Proposition 3.3;
2. $z \xleftrightarrow{a,b} u \Rightarrow pACP^+ \vdash z = u$ (see in section 2.2, and for details in [5]);
3. the term systems of $pACP^+$ and $pqACP^+$ are same, including the same operators and the same axiom systems, but different atomic action meanings (semantics).

So, we get $\langle z, \varrho \rangle \xleftrightarrow{a,b}_q \langle u, \varrho \rangle \Rightarrow pqACP^+ \vdash z = u$, as desired. □

In the following, for the existence of τ action, the quantum state ϱ is adjust to the state of all public quantum variables. Because of quantum entanglement, the entangled state that τ action affects are all in internal variables. This treatment of quantum entanglement is implicit, in section 3.3, we will introduce an explicit treatment of quantum entanglement. With the adjustment of ϱ , there is $\tau(\varrho) = \varrho$. So, for silent action τ , without consideration of any probability, there should be an action transition rule as follows.

$$\overline{\langle \tau, \varrho \rangle \xrightarrow{\tau} \langle \sqrt{\cdot}, \tau(\varrho) \rangle}$$

where $\tau(\varrho) = \varrho$ with the states of all public quantum variables ϱ .

Definition 3.11. A fully probabilistic quantum graph g is a tuple $(S_p \cup S_n \cup \{NIL\}, \rightsquigarrow, \rightarrow, \mu, \varrho, root)$ consisting of:

- a countable set S_p of probabilistic states,
- a countable set S_n of action states such that $S_p \cap S_n = \emptyset$ and $NIL \notin S_p \cup S_n$,
- $root \in S_p$,
- a relation $\rightsquigarrow \subseteq S_p \times S_n$,
- a function $\rightarrow: S_n \rightarrow (S_p \cup \{NIL\}) \times A_\tau$,
- a partial function $\mu: S_p \times S_n \rightarrow [0, 1]$ such that $\mu(p, n)$ is defined iff $(p, n) \in \rightsquigarrow$ for $(p, n) \in S_p \times S_n$ and $\sum_{n \in S_n} \mu(p, n) = 1$ for any $p \in S_p$.
- a public quantum state $\varrho \in \mathcal{D}(\mathcal{H})$, and an implicit function $\mathcal{D}(\mathcal{H}) \times A_\tau \rightarrow \mathcal{D}(\mathcal{H})$.

Definition 3.12 (Probabilistic quantum branching bisimulation). Let g and h be fully probabilistic quantum graphs. If R is an equivalence relation on $S_g \cup S_h \cup \{NIL_g, NIL_h\}$ such that:

1. $(\text{root}(g), \text{root}(h)) \in R$;
2. if $(p, q) \in R$ and $p \rightsquigarrow s$ then either
 - (a) $(s, q) \in R$ or
 - (b) there are v, t such that $(p, v), (s, t) \in R$ and $q \xRightarrow{\tau^*} v \rightsquigarrow t$ or $q \xrightarrow{\tau} \cdot \xRightarrow{\tau^*} v \rightsquigarrow t$;
3. if $(p, q) \in R$ and $p \xrightarrow{\alpha} s$ with $\varrho_g \rightarrow \varrho'_g = \alpha \varrho_g \alpha^\dagger$, then either
 - (a) $\alpha = \tau$ and $(s, q) \in R$ or
 - (b) there are v, t such that $(p, v), (s, t) \in R$ and $q \xRightarrow{\tau^*} \cdot \rightsquigarrow v \xrightarrow{\alpha} t$ with $\varrho_h \rightarrow \varrho'_h = \alpha \varrho_h \alpha^\dagger$, $\varrho'_g = \varrho'_h$ iff $\varrho_g = \varrho_h$, or $q(\xrightarrow{\tau} \cdot \rightsquigarrow)^* v \xrightarrow{\alpha} t$ with $\varrho_h \rightarrow \varrho'_h = \alpha \varrho_h \alpha^\dagger$, $\varrho'_g = \varrho'_h$ iff $\varrho_g = \varrho_h$;
4. there is an equivalence relation \tilde{R} on $\text{Entry}_R(g) \cup \text{Entry}_R(h)$ such that $\tilde{R} \subseteq R$ and
 - (a) $(\text{root}(g), \text{root}(h)) \in \tilde{R}$;
 - (b) if $(p, q) \in \tilde{R}$ then for any $C \in \text{NextEntry}_{\tilde{R}}(p) \cup \text{NextEntry}_{\tilde{R}}(q)$ and for any $\alpha \in A$, $\text{Prob}_{[p]_R}(p, \tau^*, C) = \text{Prob}_{[q]_R}(q, \tau^*, C)$ and $\text{Prob}_{[p]_R}(p, \tau^* \alpha, C) = \text{Prob}_{[q]_R}(q, \tau^* \alpha, C)$;

where $\text{Prob}, \xRightarrow{\tau^*}, \text{Entry}_R(g), [p]_R$ have the same meanings as those in probabilistic branching bisimulation in section 2.2. Then (R, \tilde{R}) is a probabilistic quantum branching bisimulation relation between g and h . We write \xleftrightarrow{pqb} if there is a probabilistic quantum bisimulation R, \tilde{R} between g and h .

For additional conditions of probabilistic quantum root branching bisimulation \xleftrightarrow{pqrb} , there is a probabilistic quantum branching bisimulation (R, \tilde{R}) between g and h such that:

1. if $\text{root}(g) \rightsquigarrow p$ then there is q in h such that $\text{root}(h) \rightsquigarrow q$ and $(p, q) \in R$;
2. if $\text{root}(h) \rightsquigarrow q$ then there is p in g such that $\text{root}(g) \rightsquigarrow p$ and $(p, q) \in R$;
3. if $\text{root}(g) \rightsquigarrow p \xrightarrow{\alpha} s$ for $\alpha \in A_\tau$ with $\varrho_g \rightarrow \varrho'_g = \alpha \varrho_g \alpha^\dagger$, then there are q and t in h such that $\text{root}(h) \rightsquigarrow q \xrightarrow{\alpha} t$ with $\varrho_h \rightarrow \varrho'_h = \alpha \varrho_h \alpha^\dagger$, $\varrho'_g = \varrho'_h$ iff $\varrho_g = \varrho_h$, and $(p, q) \in R$ and $(s, t) \in R$;
4. if $\text{root}(h) \rightsquigarrow q \xrightarrow{\alpha} t$ for $\alpha \in A_\tau$ with $\varrho_h \rightarrow \varrho'_h = \alpha \varrho_h \alpha^\dagger$, then there are p and s in g such that $\text{root}(g) \rightsquigarrow p \xrightarrow{\alpha} s$ with $\varrho_g \rightarrow \varrho'_g = \alpha \varrho_g \alpha^\dagger$, $\varrho'_g = \varrho'_h$ iff $\varrho_g = \varrho_h$, and $(p, q) \in R$ and $(s, t) \in R$.

We obtain the following obvious but important conclusion.

Proposition 3.13. \xleftrightarrow{pqrb} implies \xleftrightarrow{prb} with respect to $fpqBPA_\tau + PVR_1 + PVR_2 + \dots$.

Proof. From the definition of \xleftrightarrow{pqrb} and the definition of \xleftrightarrow{prb} (see in section 2.2), it is obvious that \xleftrightarrow{pqrb} adds additional conditions about quantum information state ϱ into \xleftrightarrow{prb} , with respect to $fpqBPA_\tau + PVR_1 + PVR_2 + \dots$. So, let x and y be $fpqBPA_\tau + PVR_1 + PVR_2 + \dots$ terms, $\langle x, \varrho \rangle \xleftrightarrow{pqrb} \langle y, \varrho \rangle \Rightarrow x \xleftrightarrow{prb} y$, as desired. \square

Theorem 3.14 (Soundness of $fpqBPA_\tau + PVR_1 + PVR_2 + \dots$). G/\xleftrightarrow{pqrb} is a model of $fpqBPA_\tau + PVR_1 + PVR_2 + \dots$.

Proof. It is already proven that \xleftrightarrow{pqrb} is an equivalent and congruent relation on $\mathbb{PT}(fpqBPA_\tau + PVR_1 + PVR_2 + \dots)$ (see details in [5]), and $fpqBPA_\tau + PVR_1 + PVR_2 + \dots \vdash x = y$ then $x \xleftrightarrow{pqrb} y$ (see in section 2.2 and [5] for details), we only need to prove that the quantum information ϱ related parts are also sound.

It is sufficient to examine every axiom in the axiom systems which are added to $fpqBPA$ to obtain $fpqBPA_\tau + PVR_1 + PVR_2 + \dots$ (in Table 6) and $PVR_1 + PVR_2 + \dots$ are sound.

1. **Axioms $TI0, TI1, TI2$.** The soundness of axioms $TI0, TI1, TI2$ are straightforward.
2. **Axiom $TI4$.** For axiom $TI4$, g represents the graph of $\tau_I(p+q)$ and h represents the graph of $\tau_I(p) + \tau_I(q)$, the identity (diagonal) relation pair on g and h , (Δ_S, Δ_S) , is still the desired probabilistic quantum root branching bisimulation relating g and h , based on the probabilistic and action transition rules of alternative composition $+$.

3. **Axiom $PrTI$** For axiom $PrTI$, g represents the graph of $\tau_I(p \boxplus_\pi q)$ and h represents the graph of $\tau_I(p) \boxplus_\pi \tau_I(q)$, the identity (diagonal) relation pair on g and h , (Δ_S, Δ_S) , is still the desired probabilistic quantum root branching bisimulation relating g and h , based on the probabilistic and action transition rules of probabilistic choice operator \boxplus_π .
4. **Axiom $T1$** . For two graphs g and $g \cdot \tau$, τ does not affect the (public) quantum state ϱ for $\tau(\varrho) = \varrho$.
5. $PVR_1 + PVR_2 + \dots$. The way a probabilistic root branching bisimulation is constructed for PVR_n in [5], is still suitable for constructing a probabilistic root quantum branching bisimulation for PVR_n .

□

3.2. Quantum Measurement in Probabilistic Process Algebra

In the above section, we introduce unitary operators as the atomic actions into probabilistic process algebra. Based on quantum configuration, we establish a pure quantum probabilistic process algebra called $PQRA$, which has the same equational logic as classical probabilistic process algebra PRA , and quantum configuration-based structured operational semantics.

But, in closed quantum systems, there is another basic quantum operation – quantum measurement, besides the unitary operator. Quantum measurements have a probabilistic nature, so, if we treat quantum measurements as another kinds of atomic actions, there should be such as the following action transition rule.

$$\overline{\langle \beta, \varrho \rangle \xrightarrow{\beta} \langle \sqrt{\cdot}, \varrho_1 \rangle \boxplus_{\pi_1} \langle \sqrt{\cdot}, \varrho_2 \rangle \boxplus_{\pi_2} \dots \boxplus_{\pi_{i-1}} \langle \sqrt{\cdot}, \varrho_i \rangle}$$

where β denotes a quantum measurement, and $\beta = \sum_i \lambda_i \beta_i$, β_i denotes the projection performed on the quantum system ϱ , $\pi_i = Tr(\beta_i \varrho)$, $\varrho_i = \beta_i \varrho \beta_i / \pi_i$.

Ideally, we extend the collection of atomic actions to contain two kinds of actions: the unitary operator α and the quantum measurement β , add the above action transition rule into semantic deductive system of $PQRA$, and also design new action transition rules of quantum measurement acting on operators of $PQRA$, to obtain an almost perfect theory on closed quantum systems.

But, the above idea is unrealistic, because i in the above action transition rule is unknown and is dependent on the concrete quantum measurement performed on some concrete quantum system ϱ . It leads to unknown transitions, so, it is not positive after deduction [22] and leads to probabilistic quantum bisimulation that the deduction system induces is not a congruence. Therefore, the soundness and the completeness of $PQRA$ are lost.

In fact, quantum measurement β can not be captured explicitly in $PQRA$ and can only be processed during modeling phase in application for verifying behaviors of closed quantum systems. The behaviors of quantum measurement β can be fully modeled by the probabilistic choice operator \boxplus_π in $PQRA$, the reasons are following:

1. by Elimination Theorem of $pqACP^+$, let p be a closed $pqACP^+$ term, then there is a closed $pqBPA$ term q such that $pqACP^+ \vdash p = q$;
2. by Elimination Theorem of $pqBPA$, let p be a closed $pqBPA$ term, then there is a closed basic $pqBPA$ term q such that $pqBPA \vdash p = q$;
3. the basic terms are exactly the terms of the form

$$x \equiv x_1 \quad (x \in \mathcal{B}_+(pBPA)) \text{ or}$$

$$x \equiv x_1 \boxplus_{\pi_1} x_2 \boxplus_{\pi_2} \dots \boxplus_{\pi_{n-1}} x_n \quad (n \geq 1)$$

where $x_i \equiv \sum_{j < l_i} \alpha_{ij} \cdot t_{ij} + \sum_{k < m_i} \beta_{ik}$ for certain $\alpha_{ij}, \beta_{ik} \in A_\delta$, basic $pqBPA$ terms t_{ij} and $n, m_i, l_i \in \mathbb{N}$.

4. informally, but in general, a process in quantum computing for closed systems with quantum measurement has the following form which can be captured by the above normal form of basic $pqBPA$ terms:

$$t_1 \boxplus_{\pi_1} t_2 \boxplus_{\pi_2} \dots \boxplus_{\pi_{i-1}} t_i$$

where $\sum_i \pi_i = 1$, and $t_i \in \mathcal{B}(pBPA)$.

There is a concrete but non-trivial problem in modeling quantum measurement, which may make the whole theory fail. In a pure quantum probabilistic process algebra, it is reasonable that we assume that all probabilistic choices are caused by quantum measurements. If we model quantum measurement only by probabilistic choice operator \boxplus_π , remember that quantum measurement will make the quantum state ϱ evolve into ϱ' , this means that probabilistic action \rightsquigarrow will make ϱ change, that is, there will be such probabilistic transition rule $\frac{p \rightsquigarrow p'}{\langle p, \varrho \rangle \rightsquigarrow \langle p', \varrho' \rangle}$. We will have to make changes to the whole theory to let probabilistic choice change the quantum state ϱ . We consider the probabilistic transition rule for alternative composition $+$, it will change to the following form.

$$\frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho' \rangle, \langle y, \varrho \rangle \rightsquigarrow \langle y', \varrho'' \rangle}{\langle x + y, \varrho \rangle \rightsquigarrow \langle x' + y', \varrho' + \varrho'' \rangle}$$

or,

$$\frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho' \rangle, \langle y, \varrho \rangle \rightsquigarrow \langle y', \varrho'' \rangle}{\langle x + y, \varrho \rangle \rightsquigarrow \langle x', \varrho' \rangle + \langle y', \varrho'' \rangle}$$

Now, alternative composition $+$ is acting on quantum state ϱ or quantum configuration $\langle p, \varrho \rangle$! The signature Σ of $PQRA$ in section 3.1 must contain ϱ or $\langle p, \varrho \rangle$, which leads to the whole theory of $PQRA$ fails.

Let the following process term represent quantum measurement during modeling phase,

$$\beta_1 \cdot t_1 \boxplus_{\pi_1} \beta_2 \cdot t_2 \boxplus_{\pi_2} \cdots \boxplus_{\pi_{i-1}} \beta_i \cdot t_i$$

where $\sum_i \pi_i = 1$, $t_i \in \mathcal{B}(pBPA)$, β denotes a quantum measurement, and $\beta = \sum_i \lambda_i \beta_i$, β_i denotes the projection performed on the quantum system ϱ , $\pi_i = \text{Tr}(\beta_i \varrho)$, $\varrho_i = \beta_i \varrho \beta_i / \pi_i$.

The above term means that, firstly, we choose a projection β_i in a quantum measurement $\beta = \sum_i \lambda_i \beta_i$ probabilistically, then, we execute (perform) the projection β_i on the closed quantum system. This also adheres to the intuition on quantum mechanics.

We define B as the collection of all projections of all quantum measurements, and make the collection of atomic actions in $PQRA$ be $A \cup B \cup C$. We see that a projection $\beta_i \in B$ has the almost same semantics as a unitary operator $\alpha \in A$. So, we add the following (probabilistic and action) transition rules into those of $PQRA$.

$$\overline{\langle \beta_i, \varrho \rangle \rightsquigarrow \langle \check{\beta}_i, \varrho \rangle}$$

$$\overline{\langle \check{\beta}_i, \varrho \rangle \xrightarrow{\beta_i} \langle \surd, \varrho' \rangle}$$

Until now, $PQRA$ works again. The two main quantum operations in a closed quantum system – the unitary operator and the quantum measurement, are fully modeled in probabilistic process algebra.

3.3. Quantum Entanglement in Probabilistic Process Algebra

In section 3.1, when we introduce the silent step τ , we re-define the quantum state of all variables ϱ to the quantum state of all public quantum variables. We say that if two quantum variables are entangled, then a quantum operation performed on one variable, then state of the other quantum variable is also changed. So, the entangled states must be all the inner variables or all the public variables. Since there is exactly only one quantum operation executed, so it is an implicit manner to model quantum entanglement.

In [23], we introduced a mechanism to explicitly define quantum entanglement in open quantum systems. This mechanism is still suitable for defining quantum entanglement in closed systems. A new constant called shadow constant denoted $(\mathbb{S})_\alpha$ corresponding to a specific unitary operator or a projection, α , and also a new kind of entanglement merge \mathbb{J} , are added to $pqACP^+$ to obtain $pqACP_{QE}^+$. If there are n quantum variables entangled, they maybe be distributed in different quantum systems, with a unitary operator or a projection

No.	Axiom
<i>PrMM4</i>	$x = x + x, y = y + y \Rightarrow (x, z)[(y, w) = x \parallel w + y \parallel z + x \mid y + x \bowtie y]$
<i>EM1</i>	$\alpha \bowtie (\mathbb{S}_\alpha = \alpha$
<i>EM2</i>	$\mathbb{S}_\alpha \bowtie \alpha = \alpha$
<i>EM3</i>	$\alpha \bowtie (\mathbb{S}_\alpha \cdot y) = \alpha \cdot y$
<i>EM4</i>	$\mathbb{S}_\alpha \bowtie (\alpha \cdot y) = \alpha \cdot y$
<i>EM5</i>	$(\alpha \cdot x) \bowtie \mathbb{S}_\alpha = \alpha \cdot x$
<i>EM6</i>	$(\mathbb{S}_\alpha \cdot x) \bowtie \alpha = \alpha \cdot x$
<i>EM7</i>	$(\alpha \cdot x) \bowtie (\mathbb{S}_\alpha \cdot y) = \alpha \cdot (x \parallel y)$
<i>EM8</i>	$(\mathbb{S}_\alpha \cdot x) \bowtie (\alpha \cdot y) = \alpha \cdot (x \parallel y)$
<i>PrEM1</i>	$(x \boxplus_\pi y) \bowtie z = x \bowtie z \boxplus_\pi y \bowtie z$
<i>PrEM2</i>	$x \bowtie (y \boxplus_\pi z) = x \bowtie y \boxplus_\pi x \bowtie z$
<i>PrEM3</i>	$z = z + z \Rightarrow (x + y) \bowtie z = x \bowtie z + y \bowtie z$
<i>PrEM4</i>	$z = z + z \Rightarrow z \bowtie (x + y) = z \bowtie x + z \bowtie y$

Table 7. Axioms for quantum entanglement

$\alpha \in A \cup B$ performed on one variable, there should be one \mathbb{S}_α executed on each variable in the other $n - 1$ variables. Thus, distributed variables are all hidden behind actions. Entanglement merge happens with the interaction between α and \mathbb{S}_α , and it is modeled as a new kind parallelism mechanism. In the following, we let $\mathbb{S} \in A \cup B$.

The axiom system for the shadow constant \mathbb{S} and the entanglement merge \bowtie is shown in Table 7. Note that the axiom *PrMM4* in Table 5 is changed into the new *PrMM4* in Table 7, and the rest axioms in Table 5 still hold (we will prove the Elimination Theorem and the Conservativity Theorem of $pqACP_{QE}^+$ with respect to $pqACP^+$).

The (probabilistic and action) transition rules for constant \mathbb{S} and entanglement merge \bowtie are as follows. Note that the probabilistic transition rules for operators \parallel and \llbracket are changed into the following new ones.

$$\begin{array}{c}
\overline{\langle \mathbb{S}, \varrho \rangle \rightsquigarrow \langle \mathbb{S}, \varrho \rangle} \\
\\
\frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle, \langle y, \varrho \rangle \rightsquigarrow \langle y', \varrho \rangle}{\langle x \parallel y, \varrho \rangle \rightsquigarrow \langle x' \parallel y + y' \parallel x + x' \mid y' + x' \bowtie y', \varrho \rangle} \\
\\
\frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle, \langle y, \varrho \rangle \rightsquigarrow \langle y', \varrho \rangle}{\langle (x, z) \rrbracket \langle (y, w), \varrho \rangle \rightsquigarrow \langle x' \parallel w + y' \parallel z + x' \mid y' + x' \bowtie y', \varrho \rangle} \\
\\
\frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle, \langle y, \varrho \rangle \rightsquigarrow \langle y', \varrho \rangle}{\langle x \bowtie y, \varrho \rangle \rightsquigarrow \langle x' \bowtie y', \varrho \rangle} \\
\\
\overline{\langle \mathbb{S}, \varrho \rangle \rightarrow \langle \sqrt{\cdot}, \varrho \rangle} \\
\\
\frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle \quad \langle y, \varrho' \rangle \xrightarrow{\mathbb{S}_\alpha} \langle y', \varrho' \rangle}{\langle x \bowtie y, \varrho \rangle \xrightarrow{\alpha} \langle x' \parallel y', \varrho' \rangle} \\
\\
\frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{\cdot}, \varrho' \rangle \quad \langle y, \varrho' \rangle \xrightarrow{\mathbb{S}_\alpha} \langle y', \varrho' \rangle}{\langle x \bowtie y, \varrho \rangle \xrightarrow{\alpha} \langle y', \varrho' \rangle} \\
\\
\frac{\langle x, \varrho' \rangle \xrightarrow{\mathbb{S}_\alpha} \langle \sqrt{\cdot}, \varrho' \rangle \quad \langle y, \varrho \rangle \xrightarrow{\alpha} \langle y', \varrho' \rangle}{\langle x \bowtie y, \varrho \rangle \xrightarrow{\alpha} \langle y', \varrho' \rangle} \\
\\
\frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{\cdot}, \varrho' \rangle \quad \langle y, \varrho' \rangle \xrightarrow{\mathbb{S}_\alpha} \langle \sqrt{\cdot}, \varrho' \rangle}{\langle x \bowtie y, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{\cdot}, \varrho' \rangle}
\end{array}$$

Definition 3.15 (PDF μ for new operators). We define PDF μ for operators \parallel , \llbracket , and \bowtie as follows, and the definitions of μ for operators \llbracket , \mid and ∂_H are same as those in $pqACP^+$.

$$\mu(x \parallel y, x' \parallel y + y' \parallel x + x' \mid y' + x' \bowtie y') = \mu(x, x') \cdot \mu(y, y')$$

$$\mu((x, z) \llbracket (y, w), x' \parallel w + y' \parallel z + x' \mid y' + x' \bowtie y') = \mu(x, x') \cdot \mu(y, y')$$

$$\mu(x \bowtie y, x' \bowtie y') = \mu(x, x') \cdot \mu(y, y')$$

Proposition 3.16. μ and μ^* are well-defined on $\mathbb{PT}(pqACP_{QE}^+)$

Proof. It follows the two cases:

1. Case $t \in \mathbb{SP}(pqACP_{RN}^+)$ processes.
 - for $t \equiv s \bowtie r$, $\mu(s \bowtie r, u) = \mu(s, v) \cdot \mu(r, w)$, if $u \equiv v \bowtie w$; otherwise, $\mu(s \bowtie r, u) = 0$. $\mu(s, v)$ and $\mu(r, w)$ are defined by the inductive hypothesis, so $\mu(s \bowtie r, u)$ is defined as well;
 - for $t \equiv s \parallel r$, $\mu(s \parallel r, u) = \mu(s, v) \cdot \mu(r, w)$, if $u \equiv v \parallel r + w \parallel s + v \mid w + v \bowtie w$; otherwise, $\mu(s \parallel r, u) = 0$. $\mu(s, v)$ and $\mu(r, w)$ are defined by the inductive hypothesis, so $\mu(s \parallel r, u)$ is defined as well;
 - for $t \equiv (s, z) \llbracket (r, w)$, $\mu((s, z) \llbracket (r, w), u) = \mu(s, x) \cdot \mu(r, y)$, if $u \equiv x \parallel w + y \parallel z + x \mid y + x \bowtie y$; otherwise, $\mu((s, z) \llbracket (r, w), u) = 0$. $\mu(s, x)$ and $\mu(r, y)$ are defined by the inductive hypothesis, so $\mu((s, z) \llbracket (r, w), u)$ is defined as well.
2. Case $t \in \mathbb{DP}(pqACP_{RN}^+)$ processes. For $t \equiv s \bowtie r$, $\mu(s \bowtie r, u) = \mu(s, v) \cdot \mu(r, w)$, if $u \equiv v \bowtie w$; otherwise, $\mu(s \bowtie r, u) = 0$. $\mu(s, v) = 0$ and $\mu(r, w) = 0$ are defined by the inductive hypothesis, so $\mu(s \bowtie r, u) = 0$ is defined as well;

It is easy to check cPDF μ^* is also well-defined on $\mathbb{PT}(pqACP_{QE}^+)$, we omit it. \square

Theorem 3.17 (Elimination theorem of $pqACP_{QE}^+$). Let p be a closed $pqACP_{QE}^+$ term. Then there is a closed $pqBPA$ term such that $pqACP_{QE}^+ \vdash p = q$.

Proof. The equational logic of $pqACP_{QE}^+$ ($pqBPA$ and $pqACP^+$) is same as that of $pACP_{QE}^+$ ($pBPA$ and $pACP^+$), and the same elimination properties. We only need to treat the new case of entanglement merge \bowtie . The case of \parallel and \llbracket can be proved similarly as in elimination theorem of $pqACP^+$ just by adding the new case of \bowtie , please refer to [5].

For the new case $p = p_1 \bowtie p_2$, p_1 and p_2 are closed $pqACP_{QE}^+$ terms. By the induction, there are closed $pqBPA$ terms q_1 and q_2 such that $pqACP_{QE}^+ \vdash p_1 = q_1$ and $pqACP_{QE}^+ \vdash p_2 = q_2$. By the elimination theorem of $pqBPA$, there are basic term r_1 and r_2 such that $pqBPA \vdash q_1 = r_1$ and $pqBPA \vdash q_2 = r_2$. Then also, $pqACP_{QE}^+ \vdash p_1 = r_1$, $pqACP_{QE}^+ \vdash p_2 = r_2$, and $pqACP_{QE}^+ \vdash p_1 \bowtie p_2 = r_1 \bowtie r_2$. In the following, we prove that there is a closed $pqBPA$ term s such that $pqACP_{QE}^+ \vdash r_1 \bowtie r_2 = s$.

It is sufficient to induct on the structures of basic terms r_1 and r_2 :

- Case $r_1 \equiv \alpha \in \{A \cup B\}_\delta$ and $r_2 \equiv (\bigotimes)_\alpha \in \{A \cup B\}_\delta$. Then $pqACP_{QE}^+ \vdash r_1 \bowtie r_2 = \alpha \bowtie (\bigotimes)_\alpha = \alpha$, and α is a closed $pqBPA$ term;
- Case $r_1 \equiv (\bigotimes)_\alpha \in \{A \cup B\}_\delta$ and $r_2 \equiv \alpha \in \{A \cup B\}_\delta$. Then $pqACP_{QE}^+ \vdash r_1 \bowtie r_2 = (\bigotimes)_\alpha \bowtie \alpha = \alpha$, and α is a closed $pqBPA$ term;
- Case $r_1 \equiv \alpha \in \{A \cup B\}_\delta$ and $r_2 \equiv (\bigotimes)_\alpha \cdot r'_2, (\bigotimes)_\alpha \in \{A \cup B\}_\delta$. Then $pqACP_{QE}^+ \vdash r_1 \bowtie r_2 = \alpha \bowtie (\bigotimes)_\alpha \cdot r'_2 = \alpha \cdot r'_2$, by induction r'_2 is a closed basic term, $\alpha \cdot r'_2$ is a closed $pqBPA$ term;
- Case $r_1 \equiv (\bigotimes)_\alpha \cdot r'_1, (\bigotimes)_\alpha \in \{A \cup B\}_\delta$ and $r_2 \equiv \alpha \in \{A \cup B\}_\delta$. Then $pqACP_{QE}^+ \vdash r_1 \bowtie r_2 = (\bigotimes)_\alpha \cdot r'_1 \bowtie \alpha = \alpha \cdot r'_1$, by induction r'_1 is a closed $pqBPA$ term, $\alpha \cdot r'_1$ is a closed $pqBPA$ term;
- Case $r_1 \equiv (\bigotimes)_\alpha \in \{A \cup B\}_\delta$ and $r_2 \equiv \alpha \cdot r'_2, \alpha \in \{A \cup B\}_\delta$. Then $pqACP_{QE}^+ \vdash r_1 \bowtie r_2 = (\bigotimes)_\alpha \bowtie \alpha \cdot r'_2 = \alpha \cdot r'_2$, by induction r'_2 is a closed $pqBPA$ term, $\alpha \cdot r'_2$ is a closed $pqBPA$ term;

- Case $r_1 \equiv \alpha \cdot r'_1, \alpha \in \{A \cup B\}_\delta$ and $r_2 \equiv (\mathbb{S})_\alpha \in \{A \cup B\}_\delta$. Then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = \alpha \cdot r'_1 \not\equiv (\mathbb{S})_\alpha = \alpha \cdot r'_1$, by induction r'_1 is a closed $pqBPA$ term, $\alpha \cdot r'_1$ is a closed $pqBPA$ term;
- Case $r_1 \equiv (\mathbb{S})_\alpha \cdot r'_1, (\mathbb{S})_\alpha \in \{A \cup B\}_\delta$ and $r_2 \equiv \alpha \cdot r'_2, \alpha \in \{A \cup B\}_\delta$. Then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = (\mathbb{S})_\alpha \cdot r'_1 \not\equiv \alpha \cdot r'_2 = \alpha \cdot (r'_1 \not\equiv r'_2)$, by induction there is a closed $pqBPA$ term s such that $pqACP_{QE}^+ \vdash r'_1 \not\equiv r'_2 = s$, and $pqACP_{QE}^+ \vdash r'_1 \not\equiv r'_2 = \alpha \cdot s$ is a closed $pqBPA$ term;
- Case $r_1 \equiv \alpha \cdot r'_1, \alpha \in \{A \cup B\}_\delta$ and $r_2 \equiv (\mathbb{S})_\alpha \cdot r'_2, (\mathbb{S})_\alpha \in \{A \cup B\}_\delta$. Then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = \alpha \cdot r'_1 \not\equiv (\mathbb{S})_\alpha \cdot r'_2 = \alpha \cdot (r'_1 \not\equiv r'_2)$, by induction there is a closed $pBPA$ term s such that $pqACP_{QE}^+ \vdash r'_1 \not\equiv r'_2 = s$, and $pqACP_{QE}^+ \vdash r'_1 \not\equiv r'_2 = \alpha \cdot s$ is a closed $pqBPA$ term;
- Case $r_1 \equiv \alpha \in \{A \cup B\}_\delta$ and $r_2 \equiv r'_2 + r''_2$. Then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = \alpha \not\equiv (r'_2 + r''_2) = \alpha \not\equiv r'_2 + \alpha \not\equiv r''_2$, by induction, there are two closed $pqBPA$ terms s_1 and s_2 such that $pqACP_{QE}^+ \vdash \alpha \not\equiv r'_2 = s_1$, and $pqACP_{QE}^+ \vdash \alpha \not\equiv r''_2 = s_2$, then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = s_1 + s_2$ which is a closed $pqBPA$ term;
- Case $r_1 \equiv r'_1 + r''_1$ and $r_2 \equiv \alpha \in \{A \cup B\}_\delta$. Then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = (r'_1 + r''_1) \not\equiv \alpha = r'_1 \not\equiv \alpha + r''_1 \not\equiv \alpha$, by induction, there are two closed $pqBPA$ terms s_1 and s_2 such that $pqACP_{QE}^+ \vdash r'_1 \not\equiv \alpha = s_1$, and $pqACP_{QE}^+ \vdash r''_1 \not\equiv \alpha = s_2$, then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = s_1 + s_2$ which is a closed $pqBPA$ term;
- Case $r_1 \equiv \alpha \cdot r'_1, \alpha \in \{A \cup B\}_\delta$ and $r_2 \equiv r'_2 + r''_2$. Then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = \alpha \cdot r'_1 \not\equiv (r'_2 + r''_2) = \alpha \cdot r'_1 \not\equiv r'_2 + \alpha \cdot r'_1 \not\equiv r''_2$, by induction, there are two closed $pqBPA$ terms s_1 and s_2 such that $pqACP_{QE}^+ \vdash \alpha \cdot r'_1 \not\equiv r'_2 = s_1$, and $pqACP_{QE}^+ \vdash \alpha \cdot r'_1 \not\equiv r''_2 = s_2$, then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = s_1 + s_2$ which is a closed $pqBPA$ term;
- Case $r_1 \equiv r'_1 + r''_1$ and $r_2 \equiv \alpha \cdot r'_2, \alpha \in \{A \cup B\}_\delta$. Then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = (r'_1 + r''_1) \not\equiv \alpha \cdot r'_2 = r'_1 \not\equiv \alpha \cdot r'_2 + r''_1 \not\equiv \alpha \cdot r'_2$, by induction, there are two closed $pqBPA$ terms s_1 and s_2 such that $pqACP_{QE}^+ \vdash r'_1 \not\equiv \alpha \cdot r'_2 = s_1$, and $pqACP_{QE}^+ \vdash r''_1 \not\equiv \alpha \cdot r'_2 = s_2$, then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = s_1 + s_2$ which is a closed $pqBPA$ term;
- Case $r_1 \equiv r'_1 + r''_1$ and $r_2 \equiv r'_2 + r''_2$. Then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = (r'_1 + r''_1) \not\equiv (r'_2 + r''_2) = r'_1 \not\equiv r'_2 + r''_1 \not\equiv r''_2$, by induction, there are four closed $pqBPA$ terms s_1, s_2, s_3 and s_4 such that $pqACP_{QE}^+ \vdash r'_1 \not\equiv r'_2 = s_1, pqACP_{QE}^+ \vdash r''_1 \not\equiv r'_2 = s_2, pqACP_{QE}^+ \vdash r'_1 \not\equiv r''_2 = s_3, pqACP_{QE}^+ \vdash r''_1 \not\equiv r''_2 = s_4$, then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = s_1 + s_2 + s_3 + s_4$ which is a closed $pqBPA$ term;
- Case $r_1 \in \mathcal{B}_+$ and $r_2 \in \mathcal{B} \setminus \mathcal{B}_+$. For some $m \in \mathbb{N}, m \geq 2, r_{2i} \in \mathcal{B}_+$ and $\pi_i \in \langle 0, 1 \rangle$ with $1 \leq i \leq m, r_2 \equiv r_{21} \boxplus_{\pi_1} r_{22} \boxplus_{\pi_2} \cdots \boxplus_{\pi_{m-1}} r_{2m}$. Then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = r_1 \not\equiv (r_{21} \boxplus_{\pi_1} r_{22} \boxplus_{\pi_2} \cdots \boxplus_{\pi_{m-1}} r_{2m}) = (r_1 \not\equiv r_{21}) \boxplus_{\pi_1} (r_1 \not\equiv r_{22}) \boxplus_{\pi_2} \cdots \boxplus_{\pi_{m-1}} (r_1 \not\equiv r_{2m})$, by induction hypothesis, there are m closed $pBPA$ terms s_1, \dots, s_m , such that $pqACP_{QE}^+ \vdash r_1 \not\equiv r_{21} = s_1, \dots, pqACP_{QE}^+ \vdash r_1 \not\equiv r_{2m} = s_m$, then $pqACP_{QE}^+ \vdash s_1 \boxplus_{\pi_1} \cdots \boxplus_{\pi_{m-1}} s_m$ which is a closed $pqBPA$ term;
- Case $r_1 \in \mathcal{B} \setminus \mathcal{B}_+$ and $r_2 \in \mathcal{B}_+$. For some $m \in \mathbb{N}, m \geq 2, r_{1i} \in \mathcal{B}_+$ and $\pi_i \in \langle 0, 1 \rangle$ with $1 \leq i \leq m, r_1 \equiv r_{11} \boxplus_{\pi_1} r_{12} \boxplus_{\pi_2} \cdots \boxplus_{\pi_{m-1}} r_{1m}$. Then $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = (r_{11} \boxplus_{\pi_1} r_{12} \boxplus_{\pi_2} \cdots \boxplus_{\pi_{m-1}} r_{1m}) \not\equiv r_2 = (r_{11} \not\equiv r_2) \boxplus_{\pi_1} (r_{12} \not\equiv r_2) \boxplus_{\pi_2} \cdots \boxplus_{\pi_{m-1}} (r_{1m} \not\equiv r_2)$, by induction hypothesis, there are m closed $pBPA$ terms s_1, \dots, s_m , such that $pqACP_{QE}^+ \vdash r_{11} \not\equiv r_2 = s_1, \dots, pqACP_{QE}^+ \vdash r_{1m} \not\equiv r_2 = s_m$, then $pqACP_{QE}^+ \vdash s_1 \boxplus_{\pi_1} \cdots \boxplus_{\pi_{m-1}} s_m$ which is a closed $pqBPA$ term;
- Case $r_1, r_2 \in \mathcal{B} \setminus \mathcal{B}_+$. For some $m \in \mathbb{N}, m \geq 2, r_{1i} \in \mathcal{B}_+$ and $\pi_i \in \langle 0, 1 \rangle$ with $1 \leq i \leq m, r_1 \equiv r_{11} \boxplus_{\pi_1} r_{12} \boxplus_{\pi_2} \cdots \boxplus_{\pi_{m-1}} r_{1m}$. For some $n \in \mathbb{N}, n \geq 2, r_{2j} \in \mathcal{B}_+$ and $\rho_j \in \langle 0, 1 \rangle$ with $1 \leq j \leq n, r_2 \equiv r_{21} \boxplus_{\rho_1} r_{22} \boxplus_{\rho_2} \cdots \boxplus_{\rho_{n-1}} r_{2n}$. $pqACP_{QE}^+ \vdash r_1 \not\equiv r_2 = \boxplus_{\pi_i, \rho_j} (r_{1i} \not\equiv r_{2j})$. By the induction, there are closed $pBPA$ terms s_{ij} such that $pqACP_{QE}^+ \vdash r_{1i} \not\equiv r_{2j} = s_{ij}$, then $pqACP_{QE}^+ \vdash \boxplus_{\pi_i, \rho_j} s_{ij}$ which is a closed $pqBPA$ term.

Note that the mismatch of $\alpha \in \{A \cup B\}_\delta$ and $(\mathbb{S})_\alpha \neq \beta \in \{A \cup B\}_\delta$ in $\alpha \not\equiv \beta = \delta$, and δ is also a closed basic $pqBPA$ term. \square

Theorem 3.18 (Congruence theorem of $pqACP_{QE}^+$). \Leftrightarrow_q is a congruence relation on $\mathbb{PT}(pqACP_{QE}^+)$ with respect to the $+, \cdot, \boxplus, \llbracket, \mid, \parallel, \llbracket, \partial_H$ and $\not\equiv$ operators.

Proof. The part of the proof for the operators of $pqACP^+$ is the same as the proof of the congruence of $pqACP^+$. Here we only give the rest of the proof which concerns the operator $\not\equiv$. For the operators \llbracket and \parallel , there are some changes related operator $\not\equiv$, the proof is similarly to that of $pqACP^+$ (please refer to [5]).

Let x, y, z and w be $\mathbb{PT}(pqACP_{QE}^+)$ processes such that $\langle x, \varrho \rangle \Leftrightarrow_q \langle y, \varrho \rangle$ and $\langle z, \varrho \rangle \Leftrightarrow_q \langle w, \varrho \rangle$. So, there exist probabilistic bisimulations R_1 and R_2 such that $(\langle x, \varrho \rangle, \langle y, \varrho \rangle) \in R_1$ and $(\langle z, \varrho \rangle, \langle w, \varrho \rangle) \in R_2$. We define a relation R in the following way:

$$R = Eq(S' \cup D' \cup S \cup D \cup R_1 \cup R_2)$$

where $S = \{(\langle p \parallel q, \varrho \rangle, \langle s \parallel t, \varrho \rangle) : p, q, s, t \in \mathbb{SP}(pqACP_{QE}^+), (\langle p, \varrho \rangle, \langle s, \varrho \rangle) \in R_1, (\langle q, \varrho \rangle, \langle t, \varrho \rangle) \in R_2\}$,
 $D = \{(\langle u \parallel q + v \parallel p + u \mid v + u \bowtie v, \varrho \rangle, \langle l \parallel t + l \parallel t + l \mid k + l \bowtie k, \varrho \rangle) : p, q, s, t \in \mathbb{SP}(pqACP_{QE}^+), u, v, l, k \in \mathbb{DP}(pqACP_{QE}^+), (\langle p, \varrho \rangle, \langle s, \varrho \rangle), (\langle u, \varrho \rangle, \langle l, \varrho \rangle) \in R_1, (\langle q, \varrho \rangle, \langle t, \varrho \rangle), (\langle v, \varrho \rangle, \langle k, \varrho \rangle) \in R_2\}$,
 $S' = \{(\langle p \bowtie q, \varrho \rangle, \langle s \bowtie t, \varrho \rangle) : p, q, s, t \in \mathbb{SP}(pqACP_{QE}^+), (\langle p, \varrho \rangle, \langle s, \varrho \rangle) \in R_1, (\langle q, \varrho \rangle, \langle t, \varrho \rangle) \in R_2\}$,
 $D' = \{(\langle u \bowtie v, \varrho \rangle, \langle l \bowtie k, \varrho \rangle) : u, v, l, k \in \mathbb{DP}(pqACP_{QE}^+), (\langle u, \varrho \rangle, \langle l, \varrho \rangle) \in R_1, (\langle v, \varrho \rangle, \langle k, \varrho \rangle) \in R_2\}$.

- **EM1:** S' and D' are equivalence relation, and S', S, R_1, R_2 contain pairs of static processes relevant to R ;
- **EM2:** if $(\langle p \bowtie q, \varrho \rangle, \langle s \bowtie t, \varrho \rangle) \in S'$ and $K \in \mathbb{DP}(pqACP_{QE}^+)/D'$, then $\langle p \bowtie q, \varrho \rangle \rightsquigarrow \langle K, \varrho \rangle$ iff $\langle s \bowtie t, \varrho \rangle \rightsquigarrow \langle K, \varrho \rangle$;
- **EM3:** if $\langle p \bowtie q, \varrho \rangle \rightsquigarrow \langle K, \varrho \rangle$ for $K \in \mathbb{DP}(pqACP_{QE}^+)/D'$, then $K = [u \bowtie v]_{D'}$ for some u, v such that $\langle p, \varrho \rangle \rightsquigarrow \langle u, \varrho \rangle$ and $\langle u, \varrho \rangle \rightsquigarrow \langle v, \varrho \rangle$. From the definition of D' , $K = [u]_{R_1} \bowtie [v]_{R_2}$;
- **EM4:** since R_1, R_2, D, D' are all subsets of R and they are equivalence relations themselves, if $M \in \mathbb{DP}(pqACP_{QE}^+)/R$, then $M = \bigcup_{i_1 \in I_1} M_{i_1}^1$, $M = \bigcup_{i_2 \in I_2} M_{i_2}^2$, $M = \bigcup_{j \in J} K_j$ and $M = \bigcup_{n \in N} M_n$ for some non-empty index set I_1, I_2, J, N and some equivalence classes $M_{i_1}^1 (i_1 \in I_1)$, $M_{i_2}^2 (i_2 \in I_2)$, $K_j (j \in J)$ and $M_n (n \in N)$ of R_1, R_2, D' and D respectively.

Now suppose that $(\langle r, \varrho \rangle, \langle r_1, \varrho \rangle) \in R$ for some $r, r_1 \in \mathbb{SP}(pqACP_{QE}^+)$ and $M \in \mathbb{DP}(pqACP_{QE}^+)/R$, then:

1. if $(\langle r, \varrho \rangle, \langle r_1, \varrho \rangle) \in R_k, k = 1, 2$, then the result follows from **EM4**;
2. if $(\langle r, \varrho \rangle, \langle r_1, \varrho \rangle) \in S$, then $r \equiv p \parallel q$ and $r_1 \equiv s \parallel t$ for some $p, q, s, t \in \mathbb{SP}(pqACP_{QE}^+)$ such that $(\langle p, \varrho \rangle, \langle s, \varrho \rangle) \in R_1$ and $(\langle q, \varrho \rangle, \langle t, \varrho \rangle) \in R_2$. Then $K_j = [u_j]_{R_1}^{[p]_{R_1}} \parallel [v_j]_{R_2}^{[q]_{R_2}}$ and $\langle p, \varrho \rangle \rightsquigarrow \langle u_j, \varrho \rangle$ and $\langle q, \varrho \rangle \rightsquigarrow \langle v_j, \varrho \rangle$. Then $\mu(p \parallel q, K_j) = \mu(s \parallel t, K_j)$, $\mu(p \parallel q, M) = \mu(s \parallel t, M)$;
3. if $(\langle r, \varrho \rangle, \langle r_1, \varrho \rangle) \in S'$, then $r \equiv p \bowtie q$ and $r_1 \equiv s \bowtie t$ for some $p, q, s, t \in \mathbb{SP}(pqACP_{QE}^+)$ such that $(\langle p, \varrho \rangle, \langle s, \varrho \rangle) \in R_1$ and $(\langle q, \varrho \rangle, \langle t, \varrho \rangle) \in R_2$. According to **EM3** and **EM4**, $K_j = [u_j]_{R_1} \bowtie [v_j]_{R_2}$ and $\langle p, \varrho \rangle \rightsquigarrow \langle u_j, \varrho \rangle$ and $\langle q, \varrho \rangle \rightsquigarrow \langle v_j, \varrho \rangle$. Then $\mu(p \bowtie q, K_j) = \mu(s \bowtie t, K_j)$, $\mu(p \bowtie q, M) = \mu(s \bowtie t, M)$.

□

Theorem 3.19 (Soundness of $pqACP_{QE}^+$). Let p and q be closed $pqACP_{QE}^+$ terms. If $pqACP_{QE}^+ \vdash p = q$ then $p \leftrightarrow_q q$.

Proof. It is already proven that \leftrightarrow_q is an equivalent and congruent relation on $\mathbb{PT}(pqACP_{QE}^+)$, it is sufficient to examine every axiom in the axiom system (in Table 7) for $pqACP_{QE}^+$ is sound.

- **Axiom PrMM4.** For a relation R , suppose that $Eq(\{(u, z)[(v, w), \varrho], \langle u \parallel w + v \parallel z + u \mid v + u \bowtie v, \varrho \rangle) = R$ with $u = u + u$, $v = v + v$, and $u, v, z, w \in \mathbb{SP}(pqACP_{QE}^+)$.

By use of the probabilistic transition rules for operators \parallel, \mid, \bowtie and \bowtie , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle (u, z)[(v, w), \varrho] \rightsquigarrow \langle u' \parallel w + v' \parallel z + u' \mid v' + u' \bowtie v', \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle u \parallel w + v \parallel z + u \mid v + u \bowtie v, \varrho \rangle \rightsquigarrow \langle u' \parallel w + v' \parallel z + u' \mid v' + u' \bowtie v', \varrho \rangle}$$

In the above two probabilistic transition rules, the condition $u = u + u$ ensures that $\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle$, $\langle u, \varrho \rangle \rightsquigarrow \langle u'', \varrho \rangle$, $\langle u', \varrho \rangle \leftrightarrow_q \langle u'', \varrho \rangle$. $v = v + v$ ensures the similar things.

With the assumption $(u, z)[(v, w) = u \parallel w + v \parallel z + u \mid v + u \bowtie v$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2). Note that, for operator \bowtie , there are not action transition rules.

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom EM1.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \alpha \bowtie (\mathbb{S}_\alpha, \varrho), \langle \alpha, \varrho \rangle) = R_1$, and $Eq(\langle \check{\alpha} \bowtie (\mathbb{S}_\alpha, \varrho), \langle \check{\alpha}, \varrho \rangle) = R_2$.
By use of the probabilistic transition rules and the action transition rules for atomic action $\alpha, \mathbb{S}_\alpha$ and entanglement merge \bowtie , we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle, \langle (\mathbb{S}_\alpha, \varrho) \rightsquigarrow \langle (\mathbb{S}_\alpha, \varrho) \rangle}{\langle \alpha \bowtie (\mathbb{S}_\alpha, \varrho) \rightsquigarrow \langle \check{\alpha} \bowtie (\mathbb{S}_\alpha, \varrho) \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle (\mathbb{S}_\alpha, \varrho') \rightarrow \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \bowtie (\mathbb{S}_\alpha, \varrho) \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumptions $\alpha \bowtie (\mathbb{S}_\alpha, \varrho) = \alpha$ and $\check{\alpha} \bowtie (\mathbb{S}_\alpha, \varrho) = \check{\alpha}$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom EM2.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle (\mathbb{S}_\alpha \bowtie \alpha, \varrho), \langle \alpha, \varrho \rangle) = R_1$, and $Eq(\langle (\mathbb{S}_\alpha \bowtie \check{\alpha}, \varrho), \langle \check{\alpha}, \varrho \rangle) = R_2$.
By use of the probabilistic transition rules and the action transition rules for atomic action $\alpha, \mathbb{S}_\alpha$ and entanglement merge \bowtie , we get:

$$\frac{\langle (\mathbb{S}_\alpha, \varrho) \rightsquigarrow \langle (\mathbb{S}_\alpha, \varrho) \rangle, \langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle (\mathbb{S}_\alpha \bowtie \alpha, \varrho) \rightsquigarrow \langle (\mathbb{S}_\alpha \bowtie \check{\alpha}, \varrho) \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle (\mathbb{S}_\alpha, \varrho') \rightarrow \langle \surd, \varrho' \rangle}{\langle (\mathbb{S}_\alpha \bowtie \check{\alpha}, \varrho) \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumptions $(\mathbb{S}_\alpha \bowtie \alpha, \varrho) = \alpha$ and $(\mathbb{S}_\alpha \bowtie \check{\alpha}, \varrho) = \check{\alpha}$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom EM3.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \alpha \bowtie (\mathbb{S}_\alpha \cdot u, \varrho), \langle \alpha \cdot u, \varrho \rangle) = R_1$ with $u \in \mathbb{SP}(pqACP_{QE}^+)$, and $Eq(\langle \check{\alpha} \bowtie (\mathbb{S}_\alpha \cdot u, \varrho), \langle \check{\alpha} \cdot u, \varrho \rangle) = R_2$ with $u \in \mathbb{SP}(pqACP_{QE}^+)$.
By use of the probabilistic transition rules and the action transition rules for atomic action $\alpha, \mathbb{S}_\alpha$, sequential composition \cdot and entanglement merge \bowtie , we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle, \langle (\mathbb{S}_\alpha, \varrho) \rightsquigarrow \langle (\mathbb{S}_\alpha, \varrho) \rangle}{\langle \alpha \bowtie (\mathbb{S}_\alpha \cdot u, \varrho) \rightsquigarrow \langle \check{\alpha} \bowtie (\mathbb{S}_\alpha \cdot u, \varrho) \rangle}$$

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \cdot u, \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot u, \varrho \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle (\mathbb{S}_\alpha, \varrho') \rightarrow \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \bowtie (\mathbb{S}_\alpha \cdot u, \varrho) \xrightarrow{\alpha} \langle u, \varrho' \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot u, \varrho \rangle \xrightarrow{\alpha} \langle u, \varrho' \rangle}$$

With the assumptions $\alpha \bowtie (\mathbb{S}_\alpha \cdot u, \varrho) = \alpha \cdot u$ and $\check{\alpha} \bowtie (\mathbb{S}_\alpha \cdot u, \varrho) = \check{\alpha} \cdot u$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom EM4.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \langle \mathbb{S}_\alpha \mathbin{\dot{\vee}} \alpha \cdot u, \varrho \rangle, \langle \alpha \cdot u, \varrho \rangle \rangle) = R_1$ with $u \in \mathbb{SP}(pqACP_{QE}^+)$, and $Eq(\langle \langle \mathbb{S}_\alpha \mathbin{\dot{\vee}} \check{\alpha} \cdot u, \varrho \rangle, \langle \check{\alpha} \cdot u, \varrho \rangle \rangle) = R_2$ with $u \in \mathbb{SP}(pqACP_{QE}^+)$. By use of the probabilistic transition rules and the action transition rules for atomic action $\alpha, \mathbb{S}_\alpha$, sequential composition \cdot and entanglement merge $\mathbin{\dot{\vee}}$, we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle, \langle \mathbb{S}_\alpha, \varrho \rangle \rightsquigarrow \langle \mathbb{S}_\alpha, \varrho \rangle}{\langle \mathbb{S}_\alpha \mathbin{\dot{\vee}} \alpha \cdot u, \varrho \rangle \rightsquigarrow \langle \mathbb{S}_\alpha \mathbin{\dot{\vee}} \check{\alpha} \cdot u, \varrho \rangle}$$

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \cdot u, \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot u, \varrho \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle \mathbb{S}_\alpha, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle \mathbb{S}_\alpha \mathbin{\dot{\vee}} \check{\alpha} \cdot u, \varrho \rangle \xrightarrow{\alpha} \langle u, \varrho' \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot u, \varrho \rangle \xrightarrow{\alpha} \langle u, \varrho' \rangle}$$

With the assumptions $\mathbb{S}_\alpha \mathbin{\dot{\vee}} \alpha \cdot u = \alpha \cdot u$ and $\mathbb{S}_\alpha \mathbin{\dot{\vee}} \check{\alpha} \cdot u = \check{\alpha} \cdot u$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom EM5.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \langle \alpha \cdot u \mathbin{\dot{\vee}} \mathbb{S}_\alpha, \varrho \rangle, \langle \alpha \cdot u, \varrho \rangle \rangle) = R_1$ with $u \in \mathbb{SP}(pqACP_{QE}^+)$, and $Eq(\langle \langle \check{\alpha} \cdot u \mathbin{\dot{\vee}} \mathbb{S}_\alpha, \varrho \rangle, \langle \check{\alpha} \cdot u, \varrho \rangle \rangle) = R_2$ with $u \in \mathbb{SP}(pqACP_{QE}^+)$. By use of the probabilistic transition rules and the action transition rules for atomic action $\alpha, \mathbb{S}_\alpha$, sequential composition \cdot and entanglement merge $\mathbin{\dot{\vee}}$, we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle, \langle \mathbb{S}_\alpha, \varrho \rangle \rightsquigarrow \langle \mathbb{S}_\alpha, \varrho \rangle}{\langle \alpha \cdot u \mathbin{\dot{\vee}} \mathbb{S}_\alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot u \mathbin{\dot{\vee}} \mathbb{S}_\alpha, \varrho \rangle}$$

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \cdot u, \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot u, \varrho \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle \mathbb{S}_\alpha, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot u \mathbin{\dot{\vee}} \mathbb{S}_\alpha, \varrho \rangle \xrightarrow{\alpha} \langle u, \varrho' \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot u, \varrho \rangle \xrightarrow{\alpha} \langle u, \varrho' \rangle}$$

With the assumptions $\alpha \cdot u \mathbin{\dot{\vee}} \mathbb{S}_\alpha = \alpha \cdot u$ and $\check{\alpha} \cdot u \mathbin{\dot{\vee}} \mathbb{S}_\alpha = \check{\alpha} \cdot u$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom EM6.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \langle \mathbb{S}_\alpha \cdot u \mathbin{\dot{\vee}} \alpha, \varrho \rangle, \langle \alpha \cdot u, \varrho \rangle \rangle) = R_1$ with $u \in \mathbb{SP}(pqACP_{QE}^+)$, and $Eq(\langle \langle \mathbb{S}_\alpha \cdot u \mathbin{\dot{\vee}} \check{\alpha}, \varrho \rangle, \langle \check{\alpha} \cdot u, \varrho \rangle \rangle) = R_2$ with $u \in \mathbb{SP}(pqACP_{QE}^+)$. By use of the probabilistic transition rules and the action transition rules for atomic action $\alpha, \mathbb{S}_\alpha$, sequential composition \cdot and entanglement merge $\mathbin{\dot{\vee}}$, we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle, \langle \mathbb{S}_\alpha, \varrho \rangle \rightsquigarrow \langle \mathbb{S}_\alpha, \varrho \rangle}{\langle \mathbb{S}_\alpha \cdot u \mathbin{\dot{\vee}} \alpha, \varrho \rangle \rightsquigarrow \langle \mathbb{S}_\alpha \cdot u \mathbin{\dot{\vee}} \check{\alpha}, \varrho \rangle}$$

$$\begin{array}{c}
\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \cdot u, \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot u, \varrho \rangle} \\
\\
\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle \mathbb{S}_{\alpha}, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle \mathbb{S}_{\alpha} \cdot u \text{ } \bowtie \text{ } \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle u, \varrho' \rangle} \\
\\
\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot u, \varrho \rangle \xrightarrow{\alpha} \langle u, \varrho' \rangle}
\end{array}$$

With the assumptions $\mathbb{S}_{\alpha} \cdot u \text{ } \bowtie \text{ } \alpha = \alpha \cdot u$ and $\mathbb{S}_{\alpha} \cdot u \text{ } \bowtie \text{ } \check{\alpha} = \check{\alpha} \cdot u$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom EM7.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \alpha \cdot u \text{ } \bowtie \text{ } \mathbb{S}_{\alpha} \cdot v, \varrho \rangle, \langle \alpha \cdot (u \parallel v), \varrho \rangle) = R_1$ with $u, v \in \mathbb{SP}(pqACP_{QE}^+)$, and $Eq(\langle \check{\alpha} \cdot u \text{ } \bowtie \text{ } \mathbb{S}_{\alpha} \cdot v, \varrho \rangle, \langle \check{\alpha} \cdot (u \parallel v), \varrho \rangle) = R_2$ with $u, v \in \mathbb{SP}(pqACP_{QE}^+)$.

By use of the probabilistic transition rules and the action transition rules for atomic action $\alpha, \mathbb{S}_{\alpha}$, sequential composition \cdot and entanglement merge $\text{ } \bowtie \text{ }$, we get:

$$\begin{array}{c}
\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle, \langle \mathbb{S}_{\alpha}, \varrho \rangle \rightsquigarrow \langle \mathbb{S}_{\alpha}, \varrho \rangle}{\langle \alpha \cdot u \text{ } \bowtie \text{ } \mathbb{S}_{\alpha} \cdot v, \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot u \text{ } \bowtie \text{ } \mathbb{S}_{\alpha} \cdot v, \varrho \rangle} \\
\\
\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \cdot (u \parallel v), \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot (u \parallel v), \varrho \rangle} \\
\\
\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle \mathbb{S}_{\alpha}, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot u \text{ } \bowtie \text{ } \mathbb{S}_{\alpha} \cdot v, \varrho \rangle \xrightarrow{\alpha} \langle (u \parallel v), \varrho' \rangle} \\
\\
\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot (u \parallel v), \varrho \rangle \xrightarrow{\alpha} \langle (u \parallel v), \varrho' \rangle}
\end{array}$$

With the assumptions $\alpha \cdot u \text{ } \bowtie \text{ } \mathbb{S}_{\alpha} \cdot v = \alpha \cdot (u \parallel v)$ and $\check{\alpha} \cdot u \text{ } \bowtie \text{ } \mathbb{S}_{\alpha} \cdot v = \check{\alpha} \cdot (u \parallel v)$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom EM8.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \mathbb{S}_{\alpha} \cdot u \text{ } \bowtie \text{ } \alpha \cdot v, \varrho \rangle, \langle \alpha \cdot (u \parallel v), \varrho \rangle) = R_1$ with $u, v \in \mathbb{SP}(pqACP_{QE}^+)$, and $Eq(\langle \mathbb{S}_{\alpha} \cdot u \text{ } \bowtie \text{ } \check{\alpha} \cdot v, \varrho \rangle, \langle \check{\alpha} \cdot (u \parallel v), \varrho \rangle) = R_2$ with $u, v \in \mathbb{SP}(pqACP_{QE}^+)$.

By use of the probabilistic transition rules and the action transition rules for atomic action $\alpha, \mathbb{S}_{\alpha}$, sequential composition \cdot and entanglement merge $\text{ } \bowtie \text{ }$, we get:

$$\begin{array}{c}
\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle, \langle \mathbb{S}_{\alpha}, \varrho \rangle \rightsquigarrow \langle \mathbb{S}_{\alpha}, \varrho \rangle}{\langle \mathbb{S}_{\alpha} \cdot u \text{ } \bowtie \text{ } \alpha \cdot v, \varrho \rangle \rightsquigarrow \langle \mathbb{S}_{\alpha} \cdot u \text{ } \bowtie \text{ } \check{\alpha} \cdot v, \varrho \rangle} \\
\\
\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \cdot (u \parallel v), \varrho \rangle \rightsquigarrow \langle \check{\alpha} \cdot (u \parallel v), \varrho \rangle} \\
\\
\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle \mathbb{S}_{\alpha}, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle \mathbb{S}_{\alpha} \cdot u \text{ } \bowtie \text{ } \check{\alpha} \cdot v, \varrho \rangle \xrightarrow{\alpha} \langle (u \parallel v), \varrho' \rangle} \\
\\
\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \cdot (u \parallel v), \varrho \rangle \xrightarrow{\alpha} \langle (u \parallel v), \varrho' \rangle}
\end{array}$$

With the assumptions $\textcircled{S}_\alpha \cdot u \bowtie \alpha \cdot v = \alpha \cdot (u \parallel v)$ and $\textcircled{S}_\alpha \cdot u \bowtie \check{\alpha} \cdot v = \check{\alpha} \cdot (u \parallel v)$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom PrEM1.** For a relation R , suppose that $Eq(\langle (u \boxplus_\pi v) \bowtie w, \varrho \rangle, \langle u \bowtie w \boxplus_\pi v \bowtie w, \varrho \rangle) = R$ with $u, v, w \in \mathbb{SP}(pqACP_{QE}^+)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_π and entanglement merge \bowtie , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle (u \boxplus_\pi v) \bowtie w, \varrho \rangle \rightsquigarrow \langle u' \bowtie w, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \bowtie w \boxplus_\pi v \bowtie w, \varrho \rangle \rightsquigarrow \langle u' \bowtie w, \varrho \rangle}$$

With the assumption $(u \boxplus_\pi v) \bowtie w = u \bowtie w \boxplus_\pi v \bowtie w$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2). Note that, for probabilistic choice operator \boxplus_π , there are not action transition rules.

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom PrEM2.** For a relation R , suppose that $Eq(\langle u \bowtie (v \boxplus_\pi w), \varrho \rangle, \langle u \bowtie v \boxplus_\pi u \bowtie w, \varrho \rangle) = R$ with $u, v, w \in \mathbb{SP}(pqACP_{QE}^+)$.

By use of the probabilistic transition rules for probabilistic choice operator \boxplus_π and entanglement merge \bowtie , we get:

$$\frac{\langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle u \bowtie (v \boxplus_\pi w), \varrho \rangle \rightsquigarrow \langle u \bowtie v', \varrho \rangle}$$

$$\frac{\langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle u \bowtie v \boxplus_\pi u \bowtie w, \varrho \rangle \rightsquigarrow \langle u \bowtie v', \varrho \rangle}$$

With the assumption $u \bowtie (v \boxplus_\pi w) = u \bowtie v \boxplus_\pi u \bowtie w$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2). Note that, for probabilistic choice operator \boxplus_π , there are not action transition rules.

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom PrEM3.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle (p + q) \bowtie s, \varrho \rangle, \langle p \bowtie s + q \bowtie s, \varrho \rangle) = R_1$ with $p, q, s \in \mathbb{DP}(pqACP_{QE}^+)$, and $Eq(\langle (u + v) \bowtie w, \varrho \rangle, \langle u \bowtie w + v \bowtie w, \varrho \rangle) = R_2$ with $w = w + w$ and $u, v, w \in \mathbb{SP}(pqACP_{QE}^+)$.

By use of the probabilistic transition rules and the action transition rules for entanglement merge \bowtie , alternative operator $+$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle, \langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle}{\langle (u + v) \bowtie w, \varrho \rangle \rightsquigarrow \langle (u' + v') \bowtie w', \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle, \langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle}{\langle u \bowtie w + v \bowtie w, \varrho \rangle \rightsquigarrow \langle u' \bowtie w' + v' \bowtie w', \varrho \rangle}$$

In the above two probabilistic transition rules, the condition $w = w + w$ ensures that $\langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle$, $\langle w, \varrho \rangle \rightsquigarrow \langle w'', \varrho \rangle$, $\langle w', \varrho \rangle \leftrightarrow_q \langle w'', \varrho \rangle$.

With the assumptions $(u + v) \bowtie w = u \bowtie w + v \bowtie w$ and $(u' + v') \bowtie w' = u' \bowtie w' + v' \bowtie w'$, we get R_2 satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\begin{array}{c}
\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle s', \varrho' \rangle}{\langle (p+q) \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle p' \parallel s', \varrho' \rangle} \\
\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle s', \varrho' \rangle}{\langle p \bowtie s+q \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle p' \parallel s', \varrho' \rangle} \\
\frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle p', \varrho' \rangle}{\langle (p+q) \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle p' \parallel s', \varrho' \rangle} \\
\frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle p', \varrho' \rangle}{\langle p \bowtie s+q \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle p' \parallel s', \varrho' \rangle} \\
\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle (p+q) \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho \rangle} \\
\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle p \bowtie s+q \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho \rangle} \\
\frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle (p+q) \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho \rangle} \\
\frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle p \bowtie s+q \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho \rangle} \\
\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle (p+q) \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \\
\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle p \bowtie s+q \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \\
\frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle (p+q) \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \\
\frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle p \bowtie s+q \bowtie s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}
\end{array}$$

With the assumptions $(p+q) \bowtie s = p \bowtie s+q \bowtie s$, we get R_1 satisfies conditions 2 and 3 in the definition of \xleftrightarrow{q} (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom *PrEM4*.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle s \bowtie (p+q), \varrho \rangle, \langle s \bowtie p+s \bowtie q, \varrho \rangle) = R_1$ with $p, q, s \in \mathbb{DP}(pqACP_{QE}^+)$, and $Eq(\langle w \bowtie (u+v), \varrho \rangle, \langle w \bowtie u+w \bowtie v, \varrho \rangle) = R_2$ with $w = w+w$ and $u, v, w \in \mathbb{SP}(pqACP_{QE}^+)$.

By use of the probabilistic transition rules and the action transition rules for entanglement merge \bowtie , alternative operator $+$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle, \langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle}{\langle w \bowtie (u+v), \varrho \rangle \rightsquigarrow \langle w' \bowtie (u'+v'), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle, \langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle}{\langle w \bowtie u + w \bowtie v, \varrho \rangle \rightsquigarrow \langle w' \bowtie u' + w' \bowtie v', \varrho \rangle}$$

In the above two probabilistic transition rules, the condition $w = w + w$ ensures that $\langle w, \varrho \rangle \rightsquigarrow \langle w', \varrho \rangle$, $\langle w, \varrho \rangle \rightsquigarrow \langle w'', \varrho \rangle$, $\langle w', \varrho \rangle \xleftrightarrow{q} \langle w'', \varrho \rangle$.

With the assumptions $w \bowtie (u + v) = w \bowtie u + w \bowtie v$ and $w' \bowtie (u' + v') = w' \bowtie u' + w' \bowtie v'$, we get R_2 satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2).

$$\begin{array}{c} \frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle s', \varrho' \rangle}{\langle s \bowtie (p + q), \varrho \rangle \xrightarrow{\alpha} \langle s' \parallel p', \varrho' \rangle} \\ \frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle s', \varrho' \rangle}{\langle s \bowtie p + s \bowtie q, \varrho \rangle \xrightarrow{\alpha} \langle s' \parallel p', \varrho' \rangle} \\ \frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle p', \varrho' \rangle}{\langle s \bowtie (p + q), \varrho \rangle \xrightarrow{\alpha} \langle s' \parallel p', \varrho' \rangle} \\ \frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle p', \varrho' \rangle}{\langle s \bowtie p + s \bowtie q, \varrho \rangle \xrightarrow{\alpha} \langle s' \parallel p', \varrho' \rangle} \\ \frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle s \bowtie (p + q), \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle} \\ \frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle s \bowtie p + s \bowtie q, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle} \\ \frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle s \bowtie (p + q), \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho' \rangle} \\ \frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle s \bowtie p + s \bowtie q, \varrho \rangle \xrightarrow{\alpha} \langle s', \varrho' \rangle} \\ \frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle s \bowtie (p + q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \\ \frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle s, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle s \bowtie p + s \bowtie q, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \\ \frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle s \bowtie (p + q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \\ \frac{\langle s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \langle p, \varrho' \rangle \rightarrow \langle \surd, \varrho' \rangle}{\langle s \bowtie p + s \bowtie q, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \end{array}$$

With the assumptions $s \bowtie (p + q) = s \bowtie p + s \bowtie q$, we get R_1 satisfies conditions 2 and 3 in the definition of \xleftrightarrow{q} (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \xleftrightarrow{q}$, as desired.

□

Proposition 3.20 (Conservativity of $\mathcal{T}_{pqACP^+_{QE}}$ with respect to \mathcal{T}_{pqACP^+}). The term-deduction system $\mathcal{T}_{pqACP^+_{QE}}$ is an operationally conservative extension of the term-deduction system \mathcal{T}_{pqACP^+} .

Proof. It is sufficient to verify the following conditions:

- $\mathcal{T}_{pqACP^+_{QE}}$ is a term-deduction system in path format;
- $\mathcal{T}_{pqACP^+} \oplus \mathcal{T}_{pqACP^+_{QE}}$ is defined;
- There are no conclusions $\langle s, \varrho \rangle \xrightarrow{\alpha} \langle t, \varrho' \rangle$ or $\langle s, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{\cdot}, \varrho' \rangle$ of a rule in $\mathcal{T}_{pqACP^+_{QE}}$ such $s = x$ or $s = f(x_1, \dots, x_n)$ for some operator f of $pqACP^+$.

These conditions hold which can be trivially checked. \square

Proposition 3.21. The term-deduction system $\mathcal{T}_{pqACP^+_{QE}}$ is an operationally conservative extension up to the probabilistic quantum bisimulation of the term-deduction system \mathcal{T}_{pqACP^+} .

Proof. In the definition of (strong) probabilistic quantum bisimulation (Definition 3.2), apart from the 4th clause in Definition 3.2, probabilistic quantum bisimulation is defined in terms of predicate and relation symbols. From Proposition 3.20, for each closed $pqACP^+$ term s , its term-relation-predicate diagrams in both \mathcal{T}_{pqACP^+} and $\mathcal{T}_{pqACP^+_{QE}}$ are the same. The concepts of PDF μ and quantum state ϱ do not disturb the notion of the probabilistic quantum bisimulation defined only in terms of predicate and relation symbols. \square

Proposition 3.22 (Conservativity of $pqACP^+_{QE}$ with respect to $pqACP^+$). $pqACP^+_{QE}$ is an equationally conservative extension of $pqACP^+$, that is, if t and s are closed $pqACP^+$ terms, then $pqACP^+ \vdash t = s \Leftrightarrow pqACP^+_{QE} \vdash t = s$.

Proof. It is proven by the following three facts:

1. $\mathcal{T}_{pqACP^+_{QE}}$ is an operationally conservative extension of \mathcal{T}_{pqACP^+} up to probabilistic quantum bisimulation (by Proposition 3.21);
2. $pqACP^+$ is a complete axiomatization with respect to the bisimulation model (by Theorem 3.10);
3. $\mathcal{T}_{pqACP^+_{QE}}$ with respect to the probabilistic quantum bisimulation equivalence induces a model of $pqACP^+_{QE}$ (by Theorem 3.19).

\square

Theorem 3.23 (Completeness of $pqACP^+_{QE}$). Let z and u are closed $pqACP^+_{QE}$ terms, if $z \xleftrightarrow{q} u$, then $pqACP^+_{QE} \vdash z = u$.

Proof. It is proven based on the following two facts:

1. $pqACP^+_{QE}$ has the elimination property for $pqBPA$ (by Theorem 3.17);
2. $pqACP^+_{QE}$ is an operationally conservative extension of $pqACP^+$ (by Proposition 3.22).

\square

4. Unifying Quantum and Classical Computing

We use a quantum process configurations $\langle p, \varrho \rangle$ to represent information related to the execution of a (probabilistic) quantum process, in which p represents the structural properties of a quantum process and ϱ expresses the quantum properties of a quantum process. We have established a whole theory about quantum processes in closed quantum systems based on probabilistic process algebra, which is called *PQRA*.

In *PQRA*, the set of actions is consisted of atomic unitary operators and projections of quantum measurements, and also the deadlock δ and the silent step τ . The execution of an atomic unitary operator α and a projection β_i not only influences of the structural part p , but also changes the state of quantum variables ϱ . We still use the framework of a quantum process configuration $\langle p, \varrho \rangle$ under the situation of classical computing. In classical computing, the execution of a (classical) atomic action a only influence the structural

part p , and maintain the quantum state ϱ unchanged. Note that, this kind of actions are alike to quantum communicating actions in section 3.1, and range over the set C of quantum communicating actions. In nature, quantum communicating actions are some kind of classical actions in contrast to quantum operations, because they are unrelated to the quantum state ϱ . The difference of a quantum communicating action and a classical communicating action is that they exchange different contents, a classical communicating action exchange the classical data by value or by reference, while a quantum communicating action exchange the quantum variables only by reference. We extend the set C of quantum communicating actions to classical atomic actions (including classical communicating actions), let classical actions $a, b \in C$.

Base on the fact that a classical action a does not affect the quantum state ϱ , we can generalize classical probabilistic process algebra PRA under the framework of quantum process configuration $\langle p, \varrho \rangle$. We only take an example of $pBPA$, while $pBPA + PR$, $pACP^+$ and $fpBPA_\tau$ are omitted.

We give the probabilistic transition rules under quantum configuration for $pBPA$ as follows.

$$\begin{array}{c} \overline{\langle a, \varrho \rangle \rightsquigarrow \langle \check{a}, \varrho \rangle} \quad \overline{\langle \delta, \varrho \rangle \rightsquigarrow \langle \check{\delta}, \varrho \rangle} \\[10pt] \frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle}{\langle x \cdot y, \varrho \rangle \rightsquigarrow \langle x' \cdot y, \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle, \langle y, \varrho \rangle \rightsquigarrow \langle y', \varrho \rangle}{\langle x + y, \varrho \rangle \rightsquigarrow \langle x' + y', \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \rightsquigarrow \langle z, \varrho \rangle}{\langle x \boxplus_\pi y, \varrho \rangle \rightsquigarrow \langle z, \varrho \rangle, \langle y \boxplus_\pi x, \varrho \rangle \rightsquigarrow \langle z, \varrho \rangle} \end{array}$$

The action transition rules for $pBPA$ based on quantum configuration are as follows.

$$\begin{array}{c} \overline{\langle \check{a}, \varrho \rangle \xrightarrow{a} \langle \check{\sqrt{}}, \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{a} \langle x', \varrho \rangle}{\langle x \cdot y, \varrho \rangle \xrightarrow{a} \langle x' \cdot y, \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{a} \langle \check{\sqrt{}}, \varrho \rangle}{\langle x \cdot y, \varrho \rangle \xrightarrow{a} \langle y, \varrho \rangle} \\[10pt] \frac{\langle x, \varrho \rangle \xrightarrow{a} \langle x', \varrho \rangle}{\langle x + y, \varrho \rangle \xrightarrow{a} \langle x', \varrho \rangle, \langle y + x, \varrho \rangle \xrightarrow{a} \langle x', \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{a} \langle \check{\sqrt{}}, \varrho \rangle}{\langle x + y, \varrho \rangle \xrightarrow{a} \langle \check{\sqrt{}}, \varrho \rangle, \langle y + x, \varrho \rangle \xrightarrow{a} \langle \check{\sqrt{}}, \varrho \rangle} \end{array}$$

We can get the following conclusions naturally.

Theorem 4.1 (Soundness of $pBPA + PR$ under quantum configuration). Let x and y be $pBPA + PR$ terms. If $pBPA + PR \vdash x = y$ then $\langle x, \varrho \rangle \xleftrightarrow{q} \langle y, \varrho \rangle$.

Theorem 4.2 (AIP^- in $\mathbb{PT}^{(\infty)}(pBPA + PR)$ under quantum configuration). If for all $n \geq 1$, $\langle \Pi_n(p), \varrho \rangle \xleftrightarrow{q} \langle \Pi_n(q), \varrho \rangle$, then $\langle p, \varrho \rangle \xleftrightarrow{q} \langle q, \varrho \rangle$.

Theorem 4.3 (Soundness of $pBPA$ under quantum configuration). Let x and y be closed $pBPA$ terms. If $pBPA \vdash x = y$ then $\langle x, \varrho \rangle \xleftrightarrow{q} \langle y, \varrho \rangle$.

Theorem 4.4 (Completeness of $pBPA$ under quantum configuration). Let z and u are closed $pBPA$ terms, if $\langle z, \varrho \rangle \xleftrightarrow{q} \langle u, \varrho \rangle$, then $pBPA \vdash z = u$.

Theorem 4.5 (Soundness of $pACP^+$ under quantum configuration). Let p and q be closed $pACP^+$ terms. If $pACP^+ \vdash p = q$ then $\langle p, \varrho \rangle \xleftrightarrow{q} \langle q, \varrho \rangle$.

Theorem 4.6 (Completeness of $pACP^+$ under quantum configuration). Let z and u are closed $pACP^+$ terms, if $\langle z, \varrho \rangle \xleftrightarrow{q} \langle u, \varrho \rangle$, then $pACP^+ \vdash z = u$.

Theorem 4.7 (Soundness of $fpBPA_\tau + PVR_1 + PVR_2 + \dots$ under quantum configuration). G/\xleftrightarrow{pqrb} is a model of $fpBPA_\tau + PVR_1 + PVR_2 + \dots$.

So, $PQRA$ and PRA are unified under the framework of quantum process configuration $\langle p, \varrho \rangle$, that is, $PQRA$ and classical PRA have the same equational logics and the same semantic models based on quantum configuration.

The unification of $PQRA$ and classical PRA has an important significance, because most quantum protocols, like the famous BB84 protocol [20] and E91 protocol [21] (their verifications are shown in section 5), are mixtures of quantum information and classical information, and quantum computing and classical computing. This unification can be used widely in verification for all quantum protocols with an assumption of closed systems.

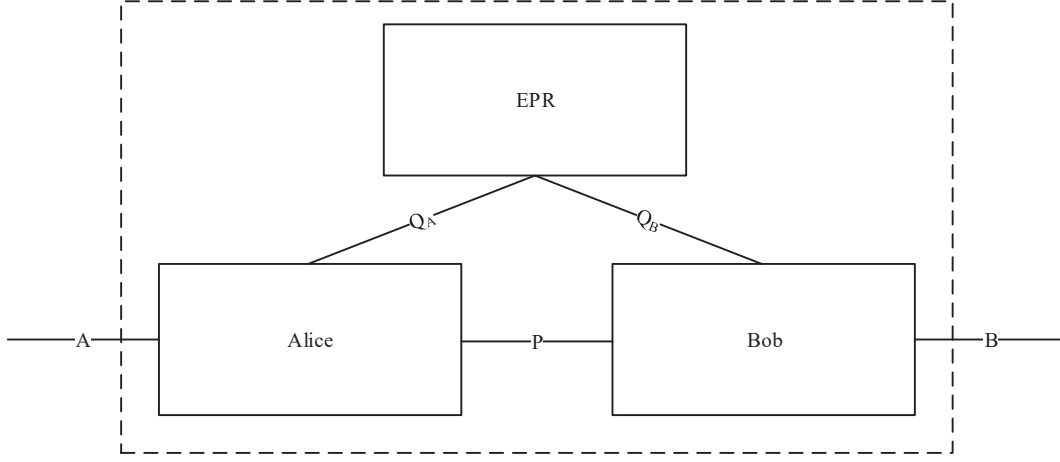


Fig. 1. Quantum teleportation protocol.

5. Applications

Quantum and classical computing in closed systems are unified with probabilistic process algebra, which have the same equational logic and the same quantum configuration based operational semantics. The unification can be used widely in verification for the behaviors of quantum and classical computing mixed systems. In this section, we show its usage in verification for the quantum communication protocols: the famous quantum teleportation [24], the first quantum communication protocol – BB84 [20], and the first quantum communication protocol using quantum entanglement – E91 [21].

5.1. Verification for Quantum Teleportation Protocol

Quantum teleportation [24] is a famous quantum protocol in quantum information theory to teleport an unknown quantum state by sending only classical information, provided that the sender and the receiver, Alice and Bob, shared an entangled state in advance. Firstly, we introduce the basic quantum teleportation protocol briefly, which is illustrated in Fig.1. In this section, we show how to process quantum entanglement in an implicit way, while in section 5.3, we show how to process quantum entanglement in an explicit way.

1. EPR generates 2-qubits entangled EPR pair $q = q_1 \otimes q_2$, and he sends q_1 to Alice through quantum channel Q_A and q_2 to Bob through quantum channel Q_B ;
2. Alice receives q_1 , after some preparations, she measures on q_1 , and sends the measurement results x to Bob through classical channel P ;
3. Bob receives q_2 from EPR, and also the classical information x from Alice. According to x , he chooses specific Pauli transformation on q_2 .

We re-introduce the basic quantum teleportation protocol in an abstract way with more technical details as Fig.1 illustrates.

Now, we assume the generation of 2-qubits q through two unitary operators $Set[q]$ and $H[q]$. EPR sends q_1 to Alice through the quantum channel Q_A by quantum communicating action $send_{Q_A}(q_1)$ and Alice receives q_1 through Q_A by quantum communicating action $receive_{Q_A}(q_1)$. Similarly, for Bob, those are $send_{Q_B}(q_2)$ and $receive_{Q_B}(q_2)$. After Alice receives q_1 , she does some preparations, including a unitary transformation $CNOT$ and a Hadamard transformation H , then Alice do measurement $M = \sum_{i=0}^3 M_i$, and sends measurement results x to Bob through the public classical channel P by classical communicating action $send_P(x)$, and Bob receives x through channel P by classical communicating action $receive_P(x)$.

According to x , Bob performs specific Pauli transformations σ_x on q_2 . Let Alice, Bob and EPR be a system ABE and let interactions between Alice, Bob and EPR be internal actions. ABE receives external input D_i through channel A by communicating action $receive_A(D_i)$ and sends results D_o through channel B by communicating action $send_B(D_o)$. Note that the entangled EPR pair $q = q_1 \otimes q_2$ is within ABE , so quantum entanglement can be processed implicitly.

Then the state transitions of EPR can be described by PQRA as follows.

$$\begin{aligned} E &= Set[q] \cdot E_1 \\ E_1 &= H[q] \cdot E_2 \\ E_2 &= send_{Q_A}(q_1) \cdot E_3 \\ E_3 &= send_{Q_B}(q_2) \cdot E \end{aligned}$$

And the state transitions of Alice can be described by PQRA as follows.

$$\begin{aligned} A &= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot A_1 \\ A_1 &= receive_{Q_A}(q_1) \cdot A_2 \\ A_2 &= CNOT \cdot A_3 \\ A_3 &= H \cdot A_4 \\ A_4 &= (M_0 \cdot send_P(0) \boxplus_{\frac{1}{4}} M_1 \cdot send_P(1) \boxplus_{\frac{1}{4}} M_2 \cdot send_P(2) \boxplus_{\frac{1}{4}} M_3 \cdot send_P(3)) \cdot A \end{aligned}$$

where Δ_i is the collection of the input data.

And the state transitions of Bob can be described by PQRA as follows.

$$\begin{aligned} B &= receive_{Q_B}(q_2) \cdot B_1 \\ B_1 &= (receive_P(0) \cdot \sigma_0 \boxplus_{\frac{1}{4}} receive_P(1) \cdot \sigma_1 \boxplus_{\frac{1}{4}} receive_P(2) \cdot \sigma_2 \boxplus_{\frac{1}{4}} receive_P(3) \cdot \sigma_3) \cdot B_2 \\ B_2 &= \sum_{D_o \in \Delta_o} send_B(D_o) \cdot B \end{aligned}$$

where Δ_o is the collection of the output data.

The send action and receive action of the same data through the same channel can communicate each other, otherwise, a deadlock δ will be caused. We define the following communication functions.

$$\begin{aligned} \gamma(send_{Q_A}(q_1), receive_{Q_A}(q_1)) &\triangleq c_{Q_A}(q_1) \\ \gamma(send_{Q_B}(q_2), receive_{Q_B}(q_2)) &\triangleq c_{Q_B}(q_2) \\ \gamma(send_P(0), receive_P(0)) &\triangleq c_P(0) \\ \gamma(send_P(1), receive_P(1)) &\triangleq c_P(1) \\ \gamma(send_P(2), receive_P(2)) &\triangleq c_P(2) \\ \gamma(send_P(3), receive_P(3)) &\triangleq c_P(3) \end{aligned}$$

Let A , B and E in parallel, then the system ABE can be represented by the following process term.

$$\tau_I(\partial_H(A \parallel B \parallel E))$$

where $H = \{send_{Q_A}(q_1), receive_{Q_A}(q_1), send_{Q_B}(q_2), receive_{Q_B}(q_2), send_P(0), receive_P(0), send_P(1), receive_P(1), send_P(2), receive_P(2), send_P(3), receive_P(3)\}$ and $I = \{Set[q], H[q], CNOT, H, M_0, M_1, M_2, M_3, \sigma_0, \sigma_1, \sigma_2, \sigma_3, c_{Q_A}(q_1), c_{Q_B}(q_2), c_P(0), c_P(1), c_P(2), c_P(3)\}$.

Then we get the following conclusion.

Theorem 5.1. The basic quantum teleportation protocol $\tau_I(\partial_H(A \parallel B \parallel E))$ exhibits desired external behaviors.

Proof.

$$\begin{aligned}
\partial_H(A \parallel B \parallel E) &= \sum_{D_i \in \Delta_i} \text{receive}_A(D_i) \cdot \partial_H(A_1 \parallel B \parallel E) \\
\partial_H(A_1 \parallel B \parallel E) &= \text{Set}[q] \cdot \partial_H(A_1 \parallel B \parallel E_1) \\
\partial_H(A_1 \parallel B \parallel E_1) &= H[q] \cdot \partial_H(A_1 \parallel B \parallel E_2) \\
\partial_H(A_1 \parallel B \parallel E_2) &= c_{Q_A}(q_1) \cdot \partial_H(A_2 \parallel B \parallel E_3) \\
\partial_H(A_2 \parallel B \parallel E_3) &= CNOT \cdot \partial_H(A_3 \parallel B \parallel E_3) \\
\partial_H(A_3 \parallel B \parallel E_3) &= H \cdot \partial_H(A_4 \parallel B \parallel E_3) \\
\partial_H(A_4 \parallel B \parallel E_3) &= c_{Q_B}(q_2) \cdot \partial_H(A_4 \parallel B_1 \parallel E) \\
\partial_H(A_4 \parallel B_1 \parallel E) &= (M_0 \cdot c_P(0) \cdot \sigma_0 \boxplus_{\frac{1}{4}} M_1 \cdot c_P(1) \cdot \sigma_1 \boxplus_{\frac{1}{4}} M_2 \cdot c_P(2) \cdot \sigma_2 \boxplus_{\frac{1}{4}} M_3 \cdot c_P(3) \cdot \sigma_3) \cdot \partial_H(A \parallel B_2 \parallel E) \\
\partial_H(A \parallel B_2 \parallel E) &= \sum_{D_o \in \Delta_o} \text{send}_B(D_o) \cdot \partial_H(A \parallel B \parallel E)
\end{aligned}$$

Let $\partial_H(A \parallel B \parallel E) = \langle X_1 | E \rangle$, where E is the following guarded linear recursion specification:

$$\begin{aligned}
\{X_1 &= \sum_{D_i \in \Delta_i} \text{receive}_A(D_i) \cdot X_2, X_2 = \text{Set}[q] \cdot X_3, X_3 = H[q] \cdot X_4, \\
X_4 &= c_{Q_A}(q_1) \cdot X_5, X_5 = CNOT \cdot X_6, X_6 = H \cdot X_7, X_7 = c_{Q_B}(q_2) \cdot X_8, \\
X_8 &= (M_0 \cdot c_P(0) \cdot \sigma_0 \boxplus_{\frac{1}{4}} M_1 \cdot c_P(1) \cdot \sigma_1 \boxplus_{\frac{1}{4}} M_2 \cdot c_P(2) \cdot \sigma_2 \boxplus_{\frac{1}{4}} M_3 \cdot c_P(3) \cdot \sigma_3) \cdot X_9, \\
X_9 &= \sum_{D_o \in \Delta_o} \text{send}_B(D_o) \cdot X_1\}
\end{aligned}$$

Then we apply abstraction operator τ_I into $\langle X_1 | E \rangle$.

$$\begin{aligned}
\tau_I(\langle X_1 | E \rangle) &= \sum_{D_i \in \Delta_i} \text{receive}_A(D_i) \cdot \tau_I(\langle X_2 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} \text{receive}_A(D_i) \cdot \tau_I(\langle X_3 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} \text{receive}_A(D_i) \cdot \tau_I(\langle X_4 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} \text{receive}_A(D_i) \cdot \tau_I(\langle X_5 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} \text{receive}_A(D_i) \cdot \tau_I(\langle X_6 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} \text{receive}_A(D_i) \cdot \tau_I(\langle X_7 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} \text{receive}_A(D_i) \cdot \tau_I(\langle X_8 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} \text{receive}_A(D_i) \cdot \tau_I(\langle X_9 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} \sum_{D_o \in \Delta_o} \text{receive}_A(D_i) \cdot \text{send}_B(D_o) \cdot \tau_I(\langle X_1 | E \rangle)
\end{aligned}$$

We get $\tau_I(\langle X_1 | E \rangle) = \sum_{D_i \in \Delta_i} \sum_{D_o \in \Delta_o} \text{receive}_A(D_i) \cdot \text{send}_B(D_o) \cdot \tau_I(\langle X_1 | E \rangle)$, that is, $\tau_I(\partial_H(A \parallel B \parallel E)) = \sum_{D_i \in \Delta_i} \sum_{D_o \in \Delta_o} \text{receive}_A(D_i) \cdot \text{send}_B(D_o) \cdot \tau_I(\partial_H(A \parallel B \parallel E))$. So, the basic quantum teleportation protocol $\tau_I(\partial_H(A \parallel B \parallel E))$ exhibits desired external behaviors. \square

5.2. Verification for BB84 Protocol

The BB84 protocol [20] is used to create a private key between two parties, Alice and Bob. Firstly, we introduce the basic BB84 protocol briefly, which is illustrated in Fig.2.

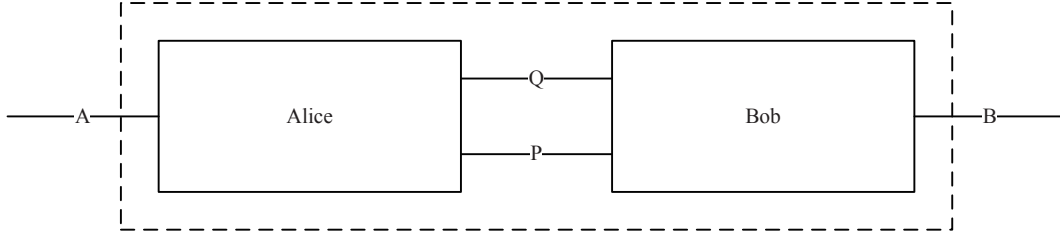


Fig. 2. BB84 protocol.

1. Alice create two string of bits with size n randomly, denoted as B_a and K_a ;
2. Alice generates a string of qubits q with size n , and the i th qubit in q is $|x_y\rangle$, where x is the i th bit of B_a and y is the i th bit of K_a ;
3. Alice sends q to Bob through a quantum channel Q between Alice and Bob;
4. Bob receives q and randomly generates a string of bits B_b with size n ;
5. Bob measures each qubit of q according to a basis by bits of B_b . And the measurement results would be K_b , which is also with size n ;
6. Bob sends his measurement bases B_b to Alice through a public channel P ;
7. Once receiving B_b , Alice sends her bases B_a to Bob through channel P , and Bob receives B_a ;
8. Alice and Bob determine that at which position the bit strings B_a and B_b are equal, and they discard the mismatched bits of B_a and B_b . Then the remaining bits of K_a and K_b , denoted as K'_a and K'_b with $K_{a,b} = K'_a = K'_b$.

We re-introduce the basic BB84 protocol in an abstract way with more technical details as Fig.2 illustrates.

Now, we assume a special measurement operation $Rand[q; B_a] = \sum_{i=0}^{2^n-1} Rand[q; B_a]_i$ which create a string of n random bits B_a from the q quantum system, and the same as $Rand[q; K_a] = \sum_{i=0}^{2^n-1} Rand[q; K_a]_i$, $Rand[q'; B_b] = \sum_{i=0}^{2^n-1} Rand[q'; B_b]_i$. $M[q; K_b] = \sum_{i=0}^{2^n-1} M[q; K_b]_i$ denotes the Bob's measurement on q . The generation of n qubits q through two unitary operators $Set_{K_a}[q]$ and $H_{B_a}[q]$. Alice sends q to Bob through the quantum channel Q by quantum communicating action $send_Q(q)$ and Bob receives q through Q by quantum communicating action $receive_Q(q)$. Bob sends B_b to Alice through the public classical channel P by classical communicating action $send_P(B_b)$ and Alice receives B_b through channel P by classical communicating action $receive_P(B_b)$, and the same as $send_P(B_a)$ and $receive_P(B_a)$. Alice and Bob generate the private key $K_{a,b}$ by a classical comparison action $cmp(K_{a,b}, K_a, K_b, B_a, B_b)$. Let Alice and Bob be a system AB and let interactions between Alice and Bob be internal actions. AB receives external input D_i through channel A by communicating action $receive_A(D_i)$ and sends results D_o through channel B by communicating action $send_B(D_o)$.

Then the state transitions of Alice can be described by PQRA as follows.

$$\begin{aligned}
A &= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot A_1 \\
A_1 &= \boxplus_{\frac{1}{2^n}, i=0}^{2^n-1} Rand[q; B_a]_i \cdot A_2 \\
A_2 &= \boxplus_{\frac{1}{2^n}, i=0}^{2^n-1} Rand[q; K_a]_i \cdot A_3 \\
A_3 &= Set_{K_a}[q] \cdot A_4 \\
A_4 &= H_{B_a}[q] \cdot A_5 \\
A_5 &= send_Q(q) \cdot A_6 \\
A_6 &= receive_P(B_b) \cdot A_7 \\
A_7 &= send_P(B_a) \cdot A_8 \\
A_8 &= cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot A
\end{aligned}$$

where Δ_i is the collection of the input data.

And the state transitions of Bob can be described by PQRA as follows.

$$\begin{aligned}
B &= receive_Q(q) \cdot B_1 \\
B_1 &= \boxplus_{\frac{1}{2n}, i=0}^{2n-1} Rand[q'; B_b]_i \cdot B_2 \\
B_2 &= \boxplus_{\frac{1}{2n}, i=0}^{2n-1} M[q; K_b]_i \cdot B_3 \\
B_3 &= send_P(B_b) \cdot B_4 \\
B_4 &= receive_P(B_a) \cdot B_5 \\
B_5 &= cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot B_6 \\
B_6 &= \sum_{D_o \in \Delta_o} send_B(D_o) \cdot B
\end{aligned}$$

where Δ_o is the collection of the output data.

The send action and receive action of the same data through the same channel can communicate each other, otherwise, a deadlock δ will be caused. We define the following communication functions.

$$\begin{aligned}
\gamma(send_Q(q), receive_Q(q)) &\triangleq c_Q(q) \\
\gamma(send_P(B_b), receive_P(B_b)) &\triangleq c_P(B_b) \\
\gamma(send_P(B_a), receive_P(B_a)) &\triangleq c_P(B_a)
\end{aligned}$$

Let A and B in parallel, then the system AB can be represented by the following process term.

$$\tau_I(\partial_H(A \parallel B))$$

where $H = \{send_Q(q), receive_Q(q), send_P(B_b), receive_P(B_b), send_P(B_a), receive_P(B_a)\}$ and $I = \{Rand[q; B_a]_i, Rand[q; K_a]_i, Set_{K_a}[q], H_{B_a}[q], Rand[q'; B_b]_i, M[q; K_b]_i, c_Q(q), c_P(B_b), c_P(B_a), cmp(K_{a,b}, K_a, K_b, B_a, B_b)\}$.

Then we get the following conclusion.

Theorem 5.2. The basic BB84 protocol $\tau_I(\partial_H(A \parallel B))$ exhibits desired external behaviors.

Proof.

$$\begin{aligned}
\partial_H(A \parallel B) &= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \partial_H(A_1 \parallel B) \\
\partial_H(A_1 \parallel B) &= \boxplus_{\frac{1}{2n}, i=0}^{2n-1} Rand[q; B_a]_i \cdot \partial_H(A_2 \parallel B) \\
\partial_H(A_2 \parallel B) &= \boxplus_{\frac{1}{2n}, i=0}^{2n-1} Rand[q; K_a]_i \cdot \partial_H(A_3 \parallel B) \\
\partial_H(A_3 \parallel B) &= Set_{K_a}[q] \cdot \partial_H(A_4 \parallel B) \\
\partial_H(A_4 \parallel B) &= H_{B_a}[q] \cdot \partial_H(A_5 \parallel B) \\
\partial_H(A_5 \parallel B) &= c_Q(q) \cdot \partial_H(A_6 \parallel B_1) \\
\partial_H(A_6 \parallel B_1) &= \boxplus_{\frac{1}{2n}, i=0}^{2n-1} Rand[q'; B_b]_i \cdot \partial_H(A_6 \parallel B_2) \\
\partial_H(A_6 \parallel B_2) &= \boxplus_{\frac{1}{2n}, i=0}^{2n-1} M[q; K_b]_i \cdot \partial_H(A_6 \parallel B_3) \\
\partial_H(A_6 \parallel B_3) &= c_P(B_b) \cdot \partial_H(A_7 \parallel B_4) \\
\partial_H(A_7 \parallel B_4) &= c_P(B_a) \cdot \partial_H(A_8 \parallel B_5) \\
\partial_H(A_8 \parallel B_5) &= cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot \partial_H(A \parallel B_5) \\
\partial_H(A \parallel B_5) &= cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot \partial_H(A \parallel B_6) \\
\partial_H(A \parallel B_6) &= \sum_{D_o \in \Delta_o} send_B(D_o) \cdot \partial_H(A \parallel B)
\end{aligned}$$

Let $\partial_H(A \parallel B) = \langle X_1 | E \rangle$, where E is the following guarded linear recursion specification:

$$\begin{aligned}
\{X_1 &= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot X_2, X_2 = \boxplus_{\frac{1}{2n}, i=0}^{2n-1} Rand[q; B_a]_i \cdot X_3, X_3 = \boxplus_{\frac{1}{2n}, i=0}^{2n-1} Rand[q; K_a]_i \cdot X_4, \\
X_4 &= Set_{K_a}[q] \cdot X_5, X_5 = H_{B_a}[q] \cdot X_6, X_6 = c_Q(q) \cdot X_7, \\
X_7 &= \boxplus_{\frac{1}{2n}, i=0}^{2n-1} Rand[q'; B_b]_i \cdot X_8, X_8 = \boxplus_{\frac{1}{2n}, i=0}^{2n-1} M[q; K_b]_i \cdot X_9, X_9 = c_P(B_b) \cdot X_{10}, X_{10} = c_P(B_a) \cdot X_{11}, \\
X_{11} &= cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot X_{12}, X_{12} = cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot X_{13}, \\
X_{13} &= \sum_{D_o \in \Delta_o} send_B(D_o) \cdot X_1\}
\end{aligned}$$

Then we apply abstraction operator τ_I into $\langle X_1 | E \rangle$.

$$\begin{aligned}
\tau_I(\langle X_1 | E \rangle) &= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_2 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_3 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_4 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_5 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_6 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_7 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_8 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_9 | E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_{10} | E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_{11} | E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_{12} | E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_{13} | E \rangle) \\
&= \sum_{D_i \in \Delta_i} \sum_{D_o \in \Delta_o} receive_A(D_i) \cdot send_B(D_o) \cdot \tau_I(\langle X_1 | E \rangle)
\end{aligned}$$

We get $\tau_I(\langle X_1 | E \rangle) = \sum_{D_i \in \Delta_i} \sum_{D_o \in \Delta_o} receive_A(D_i) \cdot send_B(D_o) \cdot \tau_I(\langle X_1 | E \rangle)$, that is, $\tau_I(\partial_H(A \parallel B)) = \sum_{D_i \in \Delta_i} \sum_{D_o \in \Delta_o} receive_A(D_i) \cdot send_B(D_o) \cdot \tau_I(\partial_H(A \parallel B))$. So, the basic BB84 protocol $\tau_I(\partial_H(A \parallel B))$ exhibits desired external behaviors. \square

5.3. Verification for E91 Protocol

With support of Entanglement merge \bowtie , PQRA can be used to verify quantum protocols utilizing entanglement explicitly. E91 protocol[21] is the first quantum protocol which utilizes entanglement. E91 protocol is used to create a private key between two parities, Alice and Bob. Firstly, we introduce the basic E91 protocol briefly, which is illustrated in Fig.3.

1. Alice generates a string of EPR pairs q with size n , i.e., $2n$ particles, and sends a string of qubits q_b from each EPR pair with n to Bob through a quantum channel Q , remains the other string of qubits q_a from each pair with size n ;

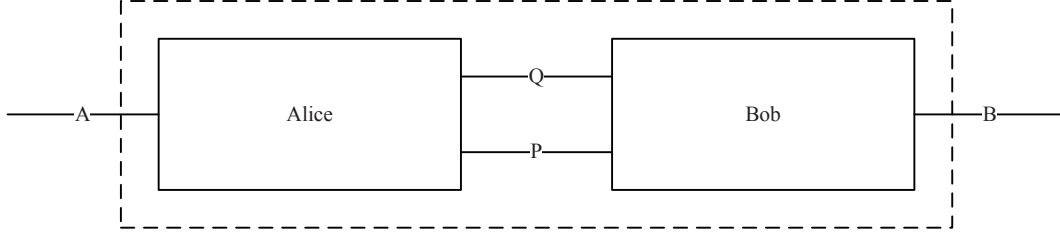


Fig. 3. E91 protocol.

2. Alice create two string of bits with size n randomly, denoted as B_a and K_a ;
3. Bob receives q_b and randomly generates a string of bits B_b with size n ;
4. Alice measures each qubit of q_a according to a basis by bits of B_a . And the measurement results would be K_a , which is also with size n ;
5. Bob measures each qubit of q_b according to a basis by bits of B_b . And the measurement results would be K_b , which is also with size n ;
6. Bob sends his measurement bases B_b to Alice through a public channel P ;
7. Once receiving B_b , Alice sends her bases B_a to Bob through channel P , and Bob receives B_a ;
8. Alice and Bob determine that at which position the bit strings B_a and B_b are equal, and they discard the mismatched bits of B_a and B_b . Then the remaining bits of K_a and K_b , denoted as K'_a and K'_b with $K_{a,b} = K'_a = K'_b$.

We re-introduce the basic E91 protocol in an abstract way with more technical details as Fig.3 illustrates.

Now, $M[q_a; K_a] = \sum_{i=0}^{2n-1} M[q_a; K_a]_i$ denotes the Alice's measurement operation of q_a , and $\textcircled{S}_{M[q_a; K_a]} = \sum_{i=0}^{2n-1} \textcircled{S}_{M[q_a; K_a]_i}$ denotes the responding shadow constant; $M[q_b; K_b] = \sum_{i=0}^{2n-1} M[q_b; K_b]_i$ denotes the Bob's measurement operation of q_b , and $\textcircled{S}_{M[q_b; K_b]} = \sum_{i=0}^{2n-1} \textcircled{S}_{M[q_b; K_b]_i}$ denotes the responding shadow constant. Alice sends q_b to Bob through the quantum channel Q by quantum communicating action $send_Q(q_b)$ and Bob receives q_b through Q by quantum communicating action $receive_Q(q_b)$. Bob sends B_b to Alice through the public channel P by classical communicating action $send_P(B_b)$ and Alice receives B_b through channel P by classical communicating action $receive_P(B_b)$, and the same as $send_P(B_a)$ and $receive_P(B_a)$. Alice and Bob generate the private key $K_{a,b}$ by a classical comparison action $cmp(K_{a,b}, K_a, K_b, B_a, B_b)$. Let Alice and Bob be a system AB and let interactions between Alice and Bob be internal actions. AB receives external input D_i through channel A by communicating action $receive_A(D_i)$ and sends results D_o through channel B by communicating action $send_B(D_o)$.

Then the state transitions of Alice can be described by PQRA as follows.

$$\begin{aligned}
 A &= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot A_1 \\
 A_1 &= send_Q(q_b) \cdot A_2 \\
 A_2 &= \boxplus_{\frac{1}{2n}, i=0}^{2n-1} M[q_a; K_a]_i \cdot A_3 \\
 A_3 &= \boxplus_{\frac{1}{2n}, i=0}^{2n-1} \textcircled{S}_{M[q_b; K_b]_i} \cdot A_4 \\
 A_4 &= receive_P(B_b) \cdot A_5 \\
 A_5 &= send_P(B_a) \cdot A_6 \\
 A_6 &= cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot A
 \end{aligned}$$

where Δ_i is the collection of the input data.

And the state transitions of Bob can be described by PQRA as follows.

$$\begin{aligned}
B &= receive_Q(q_b) \cdot B_1 \\
B_1 &= \boxplus_{\frac{1}{2^n}, i=0}^{2n-1} (\textcircled{S})_{M[q_a; K_a]_i} \cdot B_2 \\
B_2 &= \boxplus_{\frac{1}{2^n}, i=0}^{2n-1} M[q_b; K_b]_i \cdot B_3 \\
B_3 &= send_P(B_b) \cdot B_4 \\
B_4 &= receive_P(B_a) \cdot B_5 \\
B_5 &= cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot B_6 \\
B_6 &= \sum_{D_o \in \Delta_o} send_B(D_o) \cdot B
\end{aligned}$$

where Δ_o is the collection of the output data.

The send action and receive action of the same data through the same channel can communicate each other, otherwise, a deadlock δ will be caused. The quantum operation and its shadow constant pair will lead entanglement occur, otherwise, a deadlock δ will occur. We define the following communication functions.

$$\begin{aligned}
\gamma(send_Q(q_b), receive_Q(q_b)) &\triangleq c_Q(q_b) \\
\gamma(send_P(B_b), receive_P(B_b)) &\triangleq c_P(B_b) \\
\gamma(send_P(B_a), receive_P(B_a)) &\triangleq c_P(B_a)
\end{aligned}$$

Let A and B in parallel, then the system AB can be represented by the following process term.

$$\tau_I(\partial_H(A \parallel B))$$

where $H = \{send_Q(q_b), receive_Q(q_b), send_P(B_b), receive_P(B_b), send_P(B_a), receive_P(B_a), M[q_a; K_a]_i, \textcircled{S}_{M[q_a; K_a]_i}, M[q_b; K_b]_i, \textcircled{S}_{M[q_b; K_b]_i}\}$ and $I = \{c_Q(q_b), c_P(B_b), c_P(B_a), M[q_a; K_a], M[q_b; K_b], cmp(K_{a,b}, K_a, K_b, B_a, B_b)\}$.

Then we get the following conclusion.

Theorem 5.3. The basic E91 protocol $\tau_I(\partial_H(A \parallel B))$ exhibits desired external behaviors.

Proof.

$$\begin{aligned}
\partial_H(A \parallel B) &= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \partial_H(A_1 \parallel B) \\
\partial_H(A_1 \parallel B) &= c_Q(q_b) \cdot \partial_H(A_2 \parallel B_1) \\
\partial_H(A_2 \parallel B_1) &= \boxplus_{\frac{1}{2^n}, i=0}^{2n-1} M[q_a; K_a]_i \cdot \partial_H(A_3 \parallel B_2) \\
\partial_H(A_3 \parallel B_2) &= \boxplus_{\frac{1}{2^n}, i=0}^{2n-1} M[q_b; K_b]_i \cdot \partial_H(A_4 \parallel B_3) \\
\partial_H(A_4 \parallel B_3) &= c_P(B_b) \cdot \partial_H(A_5 \parallel B_4) \\
\partial_H(A_5 \parallel B_4) &= c_P(B_a) \cdot \partial_H(A_6 \parallel B_5) \\
\partial_H(A_6 \parallel B_5) &= cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot \partial_H(A \parallel B_5) \\
\partial_H(A \parallel B_5) &= cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot \partial_H(A \parallel B_6) \\
\partial_H(A \parallel B_6) &= \sum_{D_o \in \Delta_o} send_B(D_o) \cdot \partial_H(A \parallel B)
\end{aligned}$$

Let $\partial_H(A \parallel B) = \langle X_1 | E \rangle$, where E is the following guarded linear recursion specification:

$$\begin{aligned}
& \{X_1 = \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot X_2, X_2 = c_Q(q_b) \cdot X_3, \\
& X_3 = \boxplus_{\frac{1}{2^n}, i=0}^{2n-1} M[q_a; K_a]_i \cdot X_4, X_4 = \boxplus_{\frac{1}{2^n}, i=0}^{2n-1} M[q_b; K_b]_i \cdot X_5, X_5 = c_P(B_b) \cdot X_6, X_6 = c_P(B_a) \cdot X_7, \\
& X_7 = cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot X_8, X_8 = cmp(K_{a,b}, K_a, K_b, B_a, B_b) \cdot X_9, \\
& X_9 = \sum_{D_o \in \Delta_o} send_B(D_o) \cdot X_1\}
\end{aligned}$$

Then we apply abstraction operator τ_I into $\langle X_1|E \rangle$.

$$\begin{aligned}
\tau_I(\langle X_1|E \rangle) &= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_2|E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_3|E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_4|E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_5|E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_6|E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_7|E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_8|E \rangle) \\
&= \sum_{D_i \in \Delta_i} receive_A(D_i) \cdot \tau_I(\langle X_9|E \rangle) \\
&= \sum_{D_i \in \Delta_i} \sum_{D_o \in \Delta_o} receive_A(D_i) \cdot send_B(D_o) \cdot \tau_I(\langle X_1|E \rangle)
\end{aligned}$$

We get $\tau_I(\langle X_1|E \rangle) = \sum_{D_i \in \Delta_i} \sum_{D_o \in \Delta_o} receive_A(D_i) \cdot send_B(D_o) \cdot \tau_I(\langle X_1|E \rangle)$, that is, $\tau_I(\partial_H(A \parallel B)) = \sum_{D_i \in \Delta_i} \sum_{D_o \in \Delta_o} receive_A(D_i) \cdot send_B(D_o) \cdot \tau_I(\partial_H(A \parallel B))$. So, the basic E91 protocol $\tau_I(\partial_H(A \parallel B))$ exhibits desired external behaviors. \square

6. Extensions

One of the most fascinating characteristics is the modularity of ACP-like process algebra. Based on the concept of conservative extension, by introducing new operators or new constants, *PRA* and *PQRA* can have more properties. We have already seen the mechanism of extension in section 3.3.

In this section, we introducing two extensions: renaming and priorities. In these extensions, an atomic action can either be an atomic unitary operator α , a projection β_i , or be a classical atomic action a , since they are unified under the semantic model of quantum configuration. For simplicity, we only consider atomic unitary operators.

6.1. Renaming

We take an example of renaming operator $\rho_f(\alpha)$ which is used to rename the atomic actions α to $f(\alpha)$. $pACP_{RN}^+$ is obtained by extending $pACP^+$ with renaming operator ρ_f , we obtain $pqACP_{RN}^+$ make $pACP_{RN}^+$ based on quantum configuration. The axiom system for priorities is shown in Table 8.

The (probabilistic and action) transition rules for renaming operator are as follows.

No.	Axiom
RN1	$\rho_f(a) = f(a)$
RN2	$\rho_f(\delta) = \delta$
RN3	$\rho_f(x + y) = \rho_f(x) + \rho_f(y)$
RN4	$\rho_f(x \cdot y) = \rho_f(x) \cdot \rho_f(y)$
PrRN1	$\rho_f(x \boxplus_\pi y) = \rho_f(x) \boxplus_\pi \rho_f(y)$

Table 8. Axioms for renaming

$$\frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho' \rangle}{\langle \rho_f(x), \varrho \rangle \rightsquigarrow \langle \rho_f(x'), \varrho' \rangle}$$

$$\frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle}{\langle \rho_f(x), \varrho \rangle \xrightarrow{f(\alpha)} \langle \rho_f(x'), \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \rho_f(x), \varrho \rangle \xrightarrow{f(\alpha)} \langle \surd, \varrho' \rangle}$$

Definition 6.1 (PDF μ for renaming operator). We define PDF μ for operator ρ_f as follows, and the definitions of μ for other operators are same as those in $pqACP^+$.

$$\mu(\rho_f(x), \rho_f(x')) = \mu(x, x')$$

Proposition 6.2. μ and μ^* are well-defined on $\mathbb{PT}(pqACP_{RN}^+)$.

Proof. It is easy to check μ is well-defined on $\mathbb{PT}(pqACP_{RN}^+)$, it follows the two cases:

1. Case $t \in \mathbb{SP}(pqACP_{RN}^+)$ processes. For $t \equiv \rho_f(s)$, $\mu(\rho_f(s), u) = \mu(s, v)$, if $u \equiv \rho_f(v)$; otherwise, $\mu(\rho_f(s), u) = 0$. $\mu(s, v)$ is defined by the inductive hypothesis, so $\mu(\rho_f(s), u)$ is defined as well.
2. Case $t \in \mathbb{DP}(pqACP_{RN}^+)$ processes. For $t \equiv \rho_f(s)$, $\mu(\rho_f(s), u) = \mu(s, v)$, if $u \equiv \rho_f(v)$; otherwise, $\mu(\rho_f(s), u) = 0$. $\mu(s, v)$ is defined by the inductive hypothesis $\mu(s, v) = 0$, so $\mu(\rho_f(s), u) = 0$ is defined as well.

It is easy to check cPDF μ^* is also well-defined on $\mathbb{PT}(pqACP_{RN}^+)$, we omit it. \square

Theorem 6.3 (Elimination theorem of the renaming operator). Let p be a closed $pqACP_{RN}^+$ term. Then there is a closed $pqBPA$ term such that $pqACP_{RN}^+ \vdash p = q$.

Proof. The equational logic of $pqACP_{RN}^+$ ($pqBPA$ and $pqACP^+$) is same as that of $pACP_{RN}^+$ ($pBPA$ and $pACP^+$), and the same elimination properties. We only need to treat the new case of renaming operator ρ_f .

Assume that $p \equiv \rho_f(p_1)$ for certain closed $pqACP_{RN}^+$ term p_1 . By the induction hypothesis there is a closed $pqBPA$ term q_1 such that $pqACP_{RN}^+ \vdash p_1 = q_1$. Then by the elimination theorem of $pqBPA$ (see in [5]), there is a basic term r_1 such that $pqBPA \vdash q_1 = r_1$. Then, $pqACP_{RN}^+ \vdash p_1 = r_1$. By induction on the structure of basic term r_1 , we prove that there is a basic (closed) term r such that $pqACP_{RN}^+ \vdash \rho_f(r_1) = r$, and if $r_1 \in \mathcal{B}_+$ then $r \in \mathcal{B}_+$.

1. Case $r_1 \equiv \alpha$, $\alpha \in \{A \cup B\}_\delta$. Then $pqACP_{RN}^+ \vdash \rho_f(r_1) = \rho_f(\alpha) = f(\alpha)$ and $f(\alpha)$ is a basic $pqBPA$ term and $f(\alpha) \in \mathcal{B}_+$;
2. Case $r_1 \equiv \delta$. Then $pqACP_{RN}^+ \vdash \rho_f(r_1) = \rho_f(\delta) = \delta$ and δ is a basic $pqBPA$ term and $\delta \in \mathcal{B}_+$;
3. Case $r_1 = \alpha \cdot r'_1$, $\alpha \in \{A \cup B\}_\delta$ and basic term r'_1 . Then $pqACP_{RN}^+ \vdash \rho_f(r_1) = \rho_f(\alpha \cdot r'_1) = \rho_f(\alpha) \cdot \rho_f(r'_1) = f(\alpha) \cdot \rho_f(r'_1)$. By induction there is a basic term s' such that $pqACP_{RN}^+ \vdash \rho_f(r'_1) = s'$. So, $f(\alpha) \cdot s'$ is a basic term and $f(\alpha) \cdot s' \in \mathcal{B}_+$;
4. Case $r_1 = r'_1 + r''_1$ for basic \mathcal{B}_+ terms r'_1 and r''_1 . Then $pqACP_{RN}^+ \vdash \rho_f(r_1) = \rho_f(r'_1 + r''_1) = \rho_f(r'_1) + \rho_f(r''_1)$. By induction there are a basic term s' and a basic term s'' such that $pqACP_{RN}^+ \vdash \rho_f(r'_1) = s'$ and $pqACP_{RN}^+ \vdash \rho_f(r''_1) = s''$. So, $s' + s''$ is a basic term and $s' + s'' \in \mathcal{B}_+$;
5. Case $r_1 = r'_1 \boxplus_\pi r''_1$ for basic \mathcal{B}_+ terms r'_1 and r''_1 . Then $pqACP_{RN}^+ \vdash \rho_f(r_1) = \rho_f(r'_1 \boxplus_\pi r''_1) = \rho_f(r'_1) \boxplus_\pi \rho_f(r''_1)$. By induction there are a basic term s' and a basic term s'' such that $pqACP_{RN}^+ \vdash \rho_f(r'_1) = s'$ and $pqACP_{RN}^+ \vdash \rho_f(r''_1) = s''$. So, $s' \boxplus_\pi s''$ is a basic term and $s' \boxplus_\pi s'' \in \mathcal{B}_+$.

\square

Theorem 6.4 (Congruence theorem of $pqACP_{RN}^+$). \leftrightarrow_q is a congruence relation on $\mathbb{PT}(pqACP_{RN}^+)$ with respect to the $+$, \cdot , \boxplus_π , \llbracket , \lceil , \parallel , \rrbracket , ∂_H and ρ_f operators.

Proof. The part of the proof for the operators of $pqACP^+$ is the same as the proof of the congruence of $pqACP^+$. Here we only give the rest of the proof which concerns the operator ρ_f .

Let x and y be $\mathbb{PT}(pqACP_{RN}^+)$ processes such that $\langle x, \varrho \rangle \leftrightarrow_q \langle y, \varrho \rangle$. So, there exists probabilistic bisimulations R_1 such that $(\langle x, \varrho \rangle, \langle y, \varrho \rangle) \in R_1$. We define a relation R in the following way:

$$R = Eq(S \cup D \cup R_1)$$

where $S = \{(\langle \rho_f(p), \varrho \rangle, \langle \rho_f(q), \varrho \rangle) : p, q \in \mathbb{SP}(pqACP_{RN}^+), (\langle p, \varrho \rangle, \langle q, \varrho \rangle) \in R_1\}$,
 $D = \{(\langle \rho_f(u), \varrho \rangle, \langle \rho_f(v), \varrho \rangle) : u, v \in \mathbb{DP}(pqACP_{RN}^+), (\langle u, \varrho \rangle, \langle v, \varrho \rangle) \in R_1\}$.

- **RN1:** S and D are equivalence relation, and S, R_1 contain pairs of static processes relevant to R ;
- **RN2:** if $(\langle \rho_f(p), \varrho \rangle, \langle \rho_f(q), \varrho \rangle) \in S$ and $K \in \mathbb{DP}(pqACP_{RN}^+)/D$, then $\langle \rho_f(p), \varrho \rangle \rightsquigarrow \langle K, \varrho \rangle$ iff $\langle \rho_f(q), \varrho \rangle \rightsquigarrow \langle K, \varrho \rangle$;
- **RN3:** if $\langle \rho_f(p), \varrho \rangle \rightsquigarrow \langle K, \varrho \rangle$ for $K \in \mathbb{DP}(pqACP_{RN}^+)/D$, then $K = [\rho_f(u)]_D$ for some u such that $\langle p, \varrho \rangle \rightsquigarrow \langle u, \varrho \rangle$. From the definition of D , $K = \rho_f([u]_{R_1})$;
- **RN4:** since R_1, D are all subsets of R and they are equivalence relations themselves, if $M \in \mathbb{DP}(pqACP_{RN}^+)/R$, then $M = \bigcup_{i \in I} M_i$, $M = \bigcup_{j \in J} K_j$ for some non-empty index set I, J and some equivalence classes $M_i (i \in I)$, $K_j (j \in J)$ of R_1 , and D respectively.

Now suppose that $(\langle r, \varrho \rangle, \langle r_1, \varrho \rangle) \in R$ for some $r, r_1 \in \mathbb{SP}(pqACP_{RN}^+)$ and $M \in \mathbb{DP}(pqACP_{RN}^+)/R$, then:

1. if $(\langle r, \varrho \rangle, \langle r_1, \varrho \rangle) \in R_1$, then the result follows from **RN4**;
2. if $(\langle r, \varrho \rangle, \langle r_1, \varrho \rangle) \in S$, then $r \equiv \rho_f(p)$ and $r_1 \equiv \rho_f(q)$ for some $p, q \in \mathbb{SP}(pqACP_{RN}^+)$ such that $(\langle p, \varrho \rangle, \langle q, \varrho \rangle) \in R_1$. According to **RN3** and **RN4**, $K_j = \rho_f([u_j]_{R_1})$ and $\langle p, \varrho \rangle \rightsquigarrow \langle u_j, \varrho \rangle$. Then $\mu(\rho_f(p), K_j) = \mu(\rho_f(q), K_j)$, $\mu(\rho_f(p), M) = \mu(\rho_f(q), M)$.

□

Theorem 6.5 (Soundness of $pqACP_{RN}^+$). Let p and q be closed $pqACP_{RN}^+$ terms. If $pqACP_{RN}^+ \vdash p = q$ then $\langle p, \varrho \rangle \leftrightarrow_q \langle q, \varrho \rangle$.

Proof. It is already proven that \leftrightarrow_q is an equivalent and congruent relation on $\mathbb{PT}(pqACP_{RN}^+)$, it is sufficient to examine every axiom in the axiom system (in Table 8) for $pqACP_{RN}^+$ is sound.

- **Axiom RN1.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \rho_f(\alpha), \varrho \rangle, \langle f(\alpha), \varrho \rangle) = R_1$, and $Eq(\langle \rho_f(\check{\alpha}), \varrho \rangle, \langle f(\check{\alpha}), \varrho \rangle) = R_2$.

By use of the probabilistic transition rules and the action transition rules for atomic unitary operator and renaming operator ρ_f , we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \rho_f(\alpha), \varrho \rangle \rightsquigarrow \langle \rho_f(\check{\alpha}), \varrho \rangle}$$

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle f(\alpha), \varrho \rangle \rightsquigarrow \langle f(\check{\alpha}), \varrho \rangle}$$

With the assumptions $\rho_f(\alpha) = f(\alpha)$ and $\rho_f(\check{\alpha}) = f(\check{\alpha})$, we get R_1 satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \rho_f(\check{\alpha}), \varrho \rangle \xrightarrow{f(\alpha)} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle f(\check{\alpha}), \varrho \rangle \xrightarrow{f(\alpha)} \langle \surd, \varrho' \rangle}$$

With the assumption $\rho_f(\check{\alpha}) = f(\check{\alpha})$, we get R_2 satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom RN2.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \rho_f(\delta), \varrho \rangle, \langle \delta, \varrho \rangle) = R_1$, and $Eq(\langle \rho_f(\check{\delta}), \varrho \rangle, \langle \check{\delta}, \varrho \rangle) = R_2$.

By use of the probabilistic transition rules and the action transition rules for atomic unitary operator and renaming operator ρ_f , we get:

$$\frac{\langle \delta, \varrho \rangle \rightsquigarrow \langle \check{\delta}, \varrho \rangle}{\langle \rho_f(\delta), \varrho \rangle \rightsquigarrow \langle \rho_f(\check{\delta}), \varrho \rangle}$$

There are not action transition rules for the constant $\check{\delta}$, which mean that $\check{\delta}$ leads to inaction processes, accompany with the action transition rules of operator ρ_f , $\rho_f(\check{\delta}) \nrightarrow$ and $\check{\delta} \nrightarrow$ all lead to inaction processes.

With the assumptions $\rho_f(\delta) = \delta$ and $\rho_f(\check{\delta}) = \check{\delta}$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom RN3.** For a relation $R = Eq(\langle \rho_f(u+v), \varrho \rangle, \langle \rho_f(u) + \rho_f(v), \varrho \rangle) \cup Eq(\langle \rho_f(p+q), \varrho \rangle, \langle \rho_f(p) + \rho_f(q), \varrho \rangle)$, with $u, v \in \mathbb{SP}(pqACP_{RN}^+)$ and $p, q \in \mathbb{DP}(pqACP_{RN}^+)$.

By use of the probabilistic transition rules and the action transition rules for alternative operator $+$ and renaming operator ρ_f , we get:

$$\frac{\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle \rho_f(u+v), \varrho \rangle \rightsquigarrow \langle \rho_f(u') + \rho_f(v'), \varrho \rangle}}{\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle \rho_f(u) + \rho_f(v), \varrho \rangle \rightsquigarrow \langle \rho_f(u') + \rho_f(v'), \varrho \rangle}}$$

With the assumptions $\rho_f(u+v) = \rho_f(u) + \rho_f(v)$ and $\rho_f(u' + v') = \rho_f(u') + \rho_f(v')$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \rho_f(p+q), \varrho \rangle \xrightarrow{\alpha} \langle \rho_f(p'), \varrho' \rangle}}{\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \rho_f(p) + \rho_f(q), \varrho \rangle \xrightarrow{\alpha} \langle \rho_f(p'), \varrho' \rangle}}$$

$$\frac{\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \rho_f(p+q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}}{\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \rho_f(p) + \rho_f(q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}}$$

With the assumption $\rho_f(p+q) = \rho_f(p) + \rho_f(q)$, we get R satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom RN4.** For a relation $R = Eq(\langle \rho_f(u \cdot v), \varrho \rangle, \langle \rho_f(u) \cdot \rho_f(v), \varrho \rangle) \cup Eq(\langle \rho_f(p \cdot v), \varrho \rangle, \langle \rho_f(p) \cdot \rho_f(v), \varrho \rangle)$, with $u, v \in \mathbb{SP}(pqACP_{RN}^+)$ and $p \in \mathbb{DP}(pqACP_{RN}^+)$.

By use of the probabilistic transition rules and the action transition rules for sequential composition \cdot and renaming operator ρ_f , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \rho_f(u \cdot v), \varrho \rangle \rightsquigarrow \langle \rho_f(u' \cdot v), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \rho_f(u) \cdot \rho_f(v), \varrho \rangle \rightsquigarrow \langle \rho_f(u') \cdot \rho_f(v), \varrho \rangle}$$

With the assumptions $\rho_f(u \cdot v) = \rho_f(u) \cdot \rho_f(v)$ and $\rho_f(u' \cdot v) = \rho_f(u') \cdot \rho_f(v)$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \rho_f(p \cdot v), \varrho \rangle \xrightarrow{\alpha} \langle \rho_f(p' \cdot v), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \rho_f(p) \cdot \rho_f(v), \varrho \rangle \xrightarrow{\alpha} \langle \rho_f(p') \cdot \rho_f(v), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{\cdot}, \varrho' \rangle}{\langle \rho_f(p \cdot v), \varrho \rangle \xrightarrow{\alpha} \langle \rho_f(v), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \sqrt{\cdot}, \varrho' \rangle}{\langle \rho_f(p) \cdot \rho_f(v), \varrho \rangle \xrightarrow{\alpha} \langle \rho_f(v), \varrho' \rangle}$$

With the assumption $\rho_f(p \cdot v) = \rho_f(p) \cdot \rho_f(v)$ and $\rho_f(p' \cdot v) = \rho_f(p') \cdot \rho_f(v)$, we get R satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

- **Axiom $PrRN1$.** For a relation $R = Eq(\langle \rho_f(u \boxplus_{\pi} v), \varrho \rangle, \langle \rho_f(u) \boxplus_{\pi} \rho_f(v), \varrho \rangle)$, with $u, v \in \mathbb{SP}(pqACP_{RN}^+)$. By use of the probabilistic transition rules for probabilistic choice operator \boxplus_{π} and renaming operator ρ_f , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \rho_f(u \boxplus_{\pi} v), \varrho \rangle \rightsquigarrow \langle \rho_f(u'), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \rho_f(u) \boxplus_{\pi} \rho_f(v), \varrho \rangle \rightsquigarrow \langle \rho_f(u'), \varrho \rangle}$$

With the assumption $\rho_f(u \boxplus_{\pi} v) = \rho_f(u) \boxplus_{\pi} \rho_f(v)$, we get R satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

Note that, there are not action transition rules for probabilistic choice operator \boxplus_{π} .

It is easy to check that the condition on PDF μ for R also satisfies condition 4 in Definition 3.2, we omit it.

So, $R = \leftrightarrow_q$, as desired.

□

Proposition 6.6 (Conservativity of $\mathcal{T}_{pqACP_{RN}^+}$ with respect to \mathcal{T}_{pqACP^+}). The term-deduction system $\mathcal{T}_{pqACP_{RN}^+}$ is an operationally conservative extension of the term-deduction system \mathcal{T}_{pqACP^+} .

Proof. It is sufficient to verify the following conditions:

- $\mathcal{T}_{pqACP_{RN}^+}$ is a term-deduction system in path format;
- $\mathcal{T}_{pqACP^+} \oplus \mathcal{T}_{pqACP_{RN}^+}$ is defined;

No.	Axiom
TH1	$\Theta(a) = a$
TH2	$\Theta(x \cdot y) = \Theta(x) \cdot \Theta(y)$
PrTH4	$\Theta(x \boxplus_\pi y) = \Theta(x) \boxplus_\pi \Theta(y)$
DyTH3	$x = x + x, y = y + y \Rightarrow \Theta(x + y) = \Theta(x) \triangleleft y + \Theta(y) \triangleleft x$
P1	$a \triangleleft b = a \quad \text{if } \neg(a < b)$
P2	$a \triangleleft b = \delta \quad \text{if } a < b$
P3	$x \triangleleft (y \cdot z) = x \triangleleft y$
P4	$x \triangleleft (y + z) = (x \triangleleft y) \triangleleft z$
P5	$x \cdot y \triangleleft z = (x \triangleleft z) \cdot y$
P6	$(x + y) \triangleleft z = (x \triangleleft z) + (y \triangleleft z)$

Table 9. Axioms for priorities

- There are no conclusions $\langle s, \varrho \rangle \xrightarrow{\alpha} \langle t, \varrho' \rangle$ or $\langle s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle$ of a rule in $\mathcal{T}_{pqACP^+_{RN}}$ such $s = x$ or $s = f(x_1, \dots, x_n)$ for some operator f of $pqACP^+$.

These conditions hold which can be trivially checked. \square

Proposition 6.7. The term-deduction system $\mathcal{T}_{pqACP^+_{RN}}$ is an operationally conservative extension up to the probabilistic quantum bisimulation of the term-deduction system \mathcal{T}_{pqACP^+} .

Proof. In the definition of (strong) probabilistic quantum bisimulation (Definition 3.2), apart from the 4th clause in Definition 3.2, probabilistic quantum bisimulation is defined in terms of predicate and relation symbols. From Proposition 6.6, for each closed $pqACP^+$ term s , its term-relation-predicate diagrams in both \mathcal{T}_{pqACP^+} and $\mathcal{T}_{pqACP^+_{RN}}$ are the same. The concepts of PDF μ and quantum state ϱ do not disturb the notion of the probabilistic quantum bisimulation defined only in terms of predicate and relation symbols. \square

Proposition 6.8 (Conservativity of $pqACP^+_{RN}$ with respect to $pqACP^+$). $pqACP^+_{RN}$ is an equationally conservative extension of $pqACP^+$, that is, if t and s are closed $pqACP^+$ terms, then $pqACP^+ \vdash t = s \Leftrightarrow pqACP^+_{RN} \vdash t = s$.

Proof. It is proven by the following three facts:

1. $\mathcal{T}_{pqACP^+_{RN}}$ is an operationally conservative extension of \mathcal{T}_{pqACP^+} up to probabilistic quantum bisimulation (by Proposition 6.7);
2. $pqACP^+$ is a complete axiomatization with respect to the bisimulation model (by Theorem 3.10);
3. $\mathcal{T}_{pqACP^+_{RN}}$ with respect to the probabilistic quantum bisimulation equivalence induces a model of $pqACP^+_{RN}$ (by Theorem 6.5).

\square

Theorem 6.9 (Completeness of $pqACP^+_{RN}$). Let z and u are closed $pqACP^+_{RN}$ terms, if $z \xleftrightarrow{q} u$, then $pqACP^+_{RN} \vdash z = u$.

Proof. It is proven based on the following two facts:

1. $pqACP^+_{RN}$ has the elimination property for $pqBPA$ (by Theorem 6.3);
2. $pqACP^+_{RN}$ is an operationally conservative extension of $pqACP^+$ (by Proposition 6.8).

\square

6.2. Priorities

In [5], $pACP^+_{\Theta}$ is obtained by extending $pACP^+$ with two operators: the priority operator Θ and the auxiliary unless operator \triangleleft , to express priorities as a mechanism for process interruption. There is a partial order $<$ on the set of atomic actions. Corresponding to $pACP^+_{\Theta}$, we obtain $pqACP^+_{\Theta}$ make $pACP^+_{\Theta}$ based on quantum configuration. The axiom system for priorities is shown in Table 9.

The probabilistic transition rules for priorities are as follows.

$$\frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle}{\langle \Theta(x), \varrho \rangle \rightsquigarrow \langle \Theta(x'), \varrho \rangle} \quad \frac{\langle x, \varrho \rangle \rightsquigarrow \langle x', \varrho \rangle}{\langle x \triangleleft y, \varrho \rangle \rightsquigarrow \langle x' \triangleleft y, \varrho \rangle}$$

The action transition rules for priorities are as follows.

$$\begin{array}{c} \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle, \text{ and for all } b > a. \langle x, \varrho \rangle \nrightarrow^b}{\langle \Theta(x), \varrho \rangle \xrightarrow{\alpha} \langle \Theta(x'), \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \text{ and for all } b > a. \langle x, \varrho \rangle \nrightarrow^b}{\langle \Theta(x), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \\ \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle, \text{ and for all } b > a. \langle x, \varrho \rangle \nrightarrow^b}{\langle x \triangleleft y, \varrho \rangle \xrightarrow{\alpha} \langle x', \varrho' \rangle} \quad \frac{\langle x, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle, \text{ and for all } b > a. \langle x, \varrho \rangle \nrightarrow^b}{\langle x \triangleleft y, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \end{array}$$

Theorem 6.10 (Soundness of $pqACP_{\Theta}^+$). Let p and q be closed $pqACP_{\Theta}^+$ terms. If $pqACP_{\Theta}^+ \vdash p = q$ then $\langle p, \varrho \rangle \leftrightarrow_q \langle q, \varrho \rangle$.

Proof. It is already proven that \leftrightarrow_q is an equivalent and congruent relation on $\mathbb{PT}(pqACP_{\Theta}^+)$ (see details in [5]), and $pACP_{\Theta}^+ \vdash x = y$ then $x \leftrightarrow_q y$ (that is, the conditions on PDF μ are same in \leftrightarrow_q and \leftrightarrow , see in section 2.2 and [5] for details), we only need to prove that the quantum information ϱ related parts are also sound.

It is sufficient to examine every axiom in the axiom system (in Table 9) for $pqACP_{\Theta}^+$ is sound.

- **Axiom TH1.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \Theta(\alpha), \varrho \rangle, \langle \alpha, \varrho \rangle) = R_1$, and $Eq(\langle \Theta(\check{\alpha}), \varrho \rangle, \langle \check{\alpha}, \varrho \rangle) = R_2$.

By use of the probabilistic transition rules and the action transition rules for atomic unitary operator and priority operator Θ , we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \Theta(\alpha), \varrho \rangle \rightsquigarrow \langle \Theta(\check{\alpha}), \varrho \rangle} \quad \frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}$$

With the assumptions $\Theta(\alpha) = \alpha$ and $\Theta(\check{\alpha}) = \check{\alpha}$, we get R_1 satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \Theta(\check{\alpha}), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle} \quad \frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $\Theta(\check{\alpha}) = \check{\alpha}$, we get R_2 satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom TH2.** For a relation $R = Eq(\langle \Theta(u \cdot v), \varrho \rangle, \langle \Theta(u) \cdot \Theta(v), \varrho \rangle) \cup Eq(\langle \Theta(p \cdot v), \varrho \rangle, \langle \Theta(p) \cdot \Theta(v), \varrho \rangle)$, with $u, v \in \mathbb{SP}(pqACP_{\Theta}^+)$ and $p \in \mathbb{DP}(pqACP_{\Theta}^+)$.

By use of the probabilistic transition rules and the action transition rules for sequential composition \cdot and priority operator Θ , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \Theta(u \cdot v), \varrho \rangle \rightsquigarrow \langle \Theta(u') \cdot \Theta(v), \varrho \rangle} \quad \frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \Theta(u) \cdot \Theta(v), \varrho \rangle \rightsquigarrow \langle \Theta(u') \cdot \Theta(v), \varrho \rangle}$$

With the assumptions $\Theta(u \cdot v) = \Theta(u) \cdot \Theta(v)$ and $\Theta(u' \cdot v) = \Theta(u') \cdot \Theta(v)$, we get R satisfies condition 1 in the definition of \xrightarrow{q} (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \Theta(p \cdot v), \varrho \rangle \xrightarrow{\alpha} \langle \Theta(p' \cdot v), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \Theta(p) \cdot \Theta(v), \varrho \rangle \xrightarrow{\alpha} \langle \Theta(p') \cdot \Theta(v), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \Theta(p \cdot v), \varrho \rangle \xrightarrow{\alpha} \langle \Theta(v), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \Theta(p) \cdot \Theta(v), \varrho \rangle \xrightarrow{\alpha} \langle \Theta(v), \varrho' \rangle}$$

With the assumption $\Theta(p \cdot v) = \Theta(p) \cdot \Theta(v)$ and $\Theta(p' \cdot v) = \Theta(p') \cdot \Theta(v)$, we get R satisfies conditions 2 and 3 in the definition of \xrightarrow{q} (see Definition 3.2).

So, $R = \xrightarrow{q}$, as desired.

- **Axiom PrTH4.** For a relation $R = Eq(\langle \Theta(u \boxplus_{\pi} v), \varrho \rangle, \langle \Theta(u) \boxplus_{\pi} \Theta(v), \varrho \rangle)$, with $u, v \in \mathbb{SP}(pqACP_{\Theta}^+)$. By use of the probabilistic transition rules for probabilistic choice operator \boxplus_{π} and priority operator Θ , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \Theta(u \boxplus_{\pi} v), \varrho \rangle \rightsquigarrow \langle \Theta(u'), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle \Theta(u) \boxplus_{\pi} \Theta(v), \varrho \rangle \rightsquigarrow \langle \Theta(u'), \varrho \rangle}$$

With the assumption $\Theta(u \boxplus_{\pi} v) = \Theta(u) \boxplus_{\pi} \Theta(v)$, we get R satisfies condition 1 in the definition of \xrightarrow{q} (see Definition 3.2).

Note that, there are not action transition rules for probabilistic choice operator \boxplus_{π} .

So, $R = \xrightarrow{q}$, as desired.

- **Axiom DyTH3.** For a relation $R = Eq(\langle \Theta(u+v), \varrho \rangle, \langle \Theta(u) \triangleleft v + \Theta(v) \triangleleft u, \varrho \rangle) \cup Eq(\langle \Theta(p+q), \varrho \rangle, \langle \Theta(p) \triangleleft q + \Theta(q) \triangleleft p, \varrho \rangle)$, with $u, v \in \mathbb{SP}(pqACP_{\Theta}^+)$ and $p, q \in \mathbb{DP}(pqACP_{\Theta}^+)$. By use of the probabilistic transition rules and the action transition rules for alternative operator $+$, priority operator Θ and unless operator \triangleleft , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle \Theta(u+v), \varrho \rangle \rightsquigarrow \langle \Theta(u' + v'), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle \Theta(u) \triangleleft v + \Theta(v) \triangleleft u, \varrho \rangle \rightsquigarrow \langle \Theta(u') \triangleleft v' + \Theta(v') \triangleleft u', \varrho \rangle}$$

With the assumptions $\Theta(u+v) = \Theta(u) \triangleleft v + \Theta(v) \triangleleft u$ and $\Theta(u' + v') = \Theta(u') \triangleleft v' + \Theta(v') \triangleleft u'$, we get R satisfies condition 1 in the definition of \xrightarrow{q} (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \Theta(p+q), \varrho \rangle \xrightarrow{\alpha} \langle \Theta(p'), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle \Theta(p) \triangleleft q + \Theta(q) \triangleleft p, \varrho \rangle \xrightarrow{\alpha} \langle \Theta(p'), \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \Theta(p+q), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \Theta(p) \triangleleft q + \Theta(q) \triangleleft p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $\Theta(p+q) = \Theta(p) \triangleleft q + \Theta(q) \triangleleft p$, we get R satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom P1.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle \alpha \triangleleft \beta, \varrho \rangle, \langle \alpha, \varrho \rangle) = R_1$ with $\neg(\alpha < \beta)$, and $Eq(\langle \check{\alpha} \triangleleft \beta, \varrho \rangle, \langle \check{\alpha}, \varrho \rangle) = R_2$ with $\neg(\check{\alpha} < \beta)$.
By use of the probabilistic transition rules and the action transition rules for atomic unitary operator and unless operator \triangleleft , we get:

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha \triangleleft \beta, \varrho \rangle \rightsquigarrow \langle \check{\alpha} \triangleleft \beta, \varrho \rangle}$$

$$\frac{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}{\langle \alpha, \varrho \rangle \rightsquigarrow \langle \check{\alpha}, \varrho \rangle}$$

With the assumptions $\alpha \triangleleft \beta = \alpha, \neg(\alpha < \beta)$ and $\check{\alpha} \triangleleft \beta = \check{\alpha}, \neg(\check{\alpha} < \beta)$, we get R_1 satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha} \triangleleft \beta, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle \check{\alpha}, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $\check{\alpha} \triangleleft \beta = \check{\alpha}, \neg(\check{\alpha} < \beta)$, we get R_2 satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom P2.** For a relation $R = Eq(\langle \alpha \triangleleft \beta, \varrho \rangle, \langle \delta, \varrho \rangle)$ with $\alpha < \beta$.
By use of the probabilistic transition rules and the action transition rules for atomic unitary operator and unless operator \triangleleft , we get:

$$\frac{\langle \delta, \varrho \rangle \rightsquigarrow \langle \check{\delta}, \varrho \rangle}{\langle \alpha \triangleleft \beta, \varrho \rangle \rightsquigarrow \langle \check{\delta}, \varrho \rangle}$$

There are not action transition rules for the constant $\check{\delta}$, which mean that $\check{\delta}$ leads to inaction processes, accompany with the action transition rules of operator \triangleleft , $\alpha \triangleleft \beta, (\alpha < \beta) \nrightarrow$ and $\check{\delta} \nrightarrow$ all lead to inaction processes.

With the assumption $\alpha \triangleleft \beta = \delta$, we get R satisfies conditions 1, 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom P3.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle u \triangleleft (v \cdot w), \varrho \rangle, \langle u \triangleleft v, \varrho \rangle) = R_1$ with $u, v, w \in \mathbb{SP}(pqACP_{\Theta}^+)$, and $Eq(\langle p \triangleleft (q \cdot w), \varrho \rangle, \langle p \triangleleft q, \varrho \rangle) = R_2$ with $p \in \mathbb{DP}(pqACP_{\Theta}^+)$, $q, w \in \mathbb{SP}(pqACP_{\Theta}^+)$.
By use of the probabilistic transition rules and the action transition rules for atomic unitary operator α , sequential composition \cdot and unless operator \triangleleft , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \triangleleft (v \cdot w), \varrho \rangle \rightsquigarrow \langle u' \triangleleft (v \cdot w), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \triangleleft v, \varrho \rangle \rightsquigarrow \langle u' \triangleleft v, \varrho \rangle}$$

With the assumptions $u \triangleleft (v \cdot w) = u \triangleleft v$ and $u' \triangleleft (v \cdot w) = u' \triangleleft v$, we get R_1 satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p \triangleleft (q \cdot w), \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p \triangleleft q, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle p \triangleleft (q \cdot w), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle p \triangleleft q, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $p \triangleleft (q \cdot w) = p \triangleleft q$, we get R_2 satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom P4.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle u \triangleleft (v + w), \varrho \rangle, \langle u \triangleleft v \triangleleft w, \varrho \rangle) = R_1$ with $u, v, w \in \mathbb{SP}(pqACP_{\Theta}^+)$, and $Eq(\langle p \triangleleft (q + s), \varrho \rangle, \langle p \triangleleft q \triangleleft s, \varrho \rangle) = R_2$ with $p \in \mathbb{DP}(pqACP_{\Theta}^+)$, $q, s \in \mathbb{SP}(pqACP_{\Theta}^+)$. By use of the probabilistic transition rules and the action transition rules for atomic unitary operator α , alternative composition $+$ and unless operator \triangleleft , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \triangleleft (v + w), \varrho \rangle \rightsquigarrow \langle u' \triangleleft (v + w), \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle u \triangleleft v \triangleleft w, \varrho \rangle \rightsquigarrow \langle u' \triangleleft v \triangleleft w, \varrho \rangle}$$

With the assumption $u \triangleleft (v + w) = u \triangleleft v \triangleleft w$ and $u' \triangleleft (v + w) = u' \triangleleft v \triangleleft w$, we get R_1 satisfies condition 1 in the definition of \leftrightarrow_q (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p \triangleleft (q + s), \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p \triangleleft q \triangleleft s, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle p \triangleleft (q + s), \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle p \triangleleft q \triangleleft s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumption $p \triangleleft (q + s) = p \triangleleft q \triangleleft s$, we get R_2 satisfies conditions 2 and 3 in the definition of \leftrightarrow_q (see Definition 3.2).

So, $R = \leftrightarrow_q$, as desired.

- **Axiom P5.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle (u \cdot v) \triangleleft w, \varrho \rangle, \langle (u \triangleleft w) \cdot v, \varrho \rangle) = R_1$ with $u, v, w \in \mathbb{SP}(pqACP_{\Theta}^+)$, and $Eq(\langle (p \cdot q) \triangleleft s, \varrho \rangle, \langle (p \triangleleft s) \cdot q, \varrho \rangle) = R_2$ with $p \in \mathbb{DP}(pqACP_{\Theta}^+)$, $s, q \in \mathbb{SP}(pqACP_{\Theta}^+)$.

By use of the probabilistic transition rules and the action transition rules for atomic unitary operator α , sequential composition \cdot and unless operator \triangleleft , we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle (u \cdot v) \triangleleft w, \varrho \rangle \rightsquigarrow \langle (u' \cdot v) \triangleleft w, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle}{\langle (u \triangleleft w) \cdot v, \varrho \rangle \rightsquigarrow \langle (u' \triangleleft w) \cdot v, \varrho \rangle}$$

With the assumptions $(u \cdot v) \triangleleft w = (u \triangleleft w) \cdot v$ and $(u' \cdot v) \triangleleft w = (u' \triangleleft w) \cdot v$, we get R_1 satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle (p \cdot q) \triangleleft s, \varrho \rangle \xrightarrow{\alpha} \langle p' \cdot q, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle (p \triangleleft s) \cdot q, \varrho \rangle \xrightarrow{\alpha} \langle p' \cdot q, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle (p \cdot q) \triangleleft s, \varrho \rangle \xrightarrow{\alpha} \langle q, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle (p \triangleleft s) \cdot q, \varrho \rangle \xrightarrow{\alpha} \langle q, \varrho' \rangle}$$

With the assumption $(p \cdot q) \triangleleft s = (p \triangleleft s) \cdot q$, we get R_2 satisfies conditions 2 and 3 in the definition of \xleftrightarrow{q} (see Definition 3.2).

So, $R = \xleftrightarrow{q}$, as desired.

- **Axiom P6.** For a relation $R = R_1 \cup R_2$, suppose that $Eq(\langle (p + q) \triangleleft s, \varrho \rangle, \langle p \triangleleft s + q \triangleleft s, \varrho \rangle) = R_1$ with $p, q \in \mathbb{DP}(pqACP_{\Theta}^+)$ and $s \in \mathbb{SP}(pqACP_{\Theta}^+)$, and $Eq(\langle (u + v) \triangleleft w, \varrho \rangle, \langle u \triangleleft w + v \triangleleft w, \varrho \rangle) = R_2$ with $u, v, w \in \mathbb{SP}(pqACP_{\Theta}^+)$.

By use of the probabilistic transition rules and the action transition rules for unless operator \triangleleft , alternative operator $+$, we get:

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle (u + v) \triangleleft w, \varrho \rangle \rightsquigarrow \langle (u' + v') \triangleleft w, \varrho \rangle}$$

$$\frac{\langle u, \varrho \rangle \rightsquigarrow \langle u', \varrho \rangle, \langle v, \varrho \rangle \rightsquigarrow \langle v', \varrho \rangle}{\langle u \triangleleft w + v \triangleleft w, \varrho \rangle \rightsquigarrow \langle u' \triangleleft w + v' \triangleleft w, \varrho \rangle}$$

With the assumptions $(u + v) \triangleleft w = u \triangleleft w + v \triangleleft w$ and $(u' + v') \triangleleft w = u' \triangleleft w + v' \triangleleft w$, we get R_2 satisfies condition 1 in the definition of \xleftrightarrow{q} (see Definition 3.2).

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle (p + q) \triangleleft s, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}{\langle p \triangleleft s + q \triangleleft s, \varrho \rangle \xrightarrow{\alpha} \langle p', \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle (p + q) \triangleleft s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

$$\frac{\langle p, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}{\langle p \triangleleft s + q \triangleleft s, \varrho \rangle \xrightarrow{\alpha} \langle \surd, \varrho' \rangle}$$

With the assumptions $(p + q) \triangleleft s = p \triangleleft s + q \triangleleft s$, we get R_1 satisfies conditions 2 and 3 in the definition of \xrightarrow{q} (see Definition 3.2).

So, $R = \xrightarrow{q}$, as desired.

□

Theorem 6.11 (Completeness of $pqACP_{\Theta}^+$). Let z and u are closed $pqACP_{\Theta}^+$ terms, if $\langle z, \varrho \rangle \xrightarrow{q} \langle u, \varrho \rangle$, then $pqACP_{\Theta}^+ \vdash z = u$.

Proof. It is based on the following three facts: let z and u are closed $pqACP_{\Theta}^+$ terms,

1. $\langle z, \varrho \rangle \xrightarrow{q} \langle u, \varrho \rangle \Rightarrow z \xleftrightarrow{q} u$ by Proposition 3.3;
2. $z \xleftrightarrow{q} u \Rightarrow pACP_{\Theta}^+ \vdash z = u$ (see in [5]);
3. the term systems of $pACP_{\Theta}^+$ and $pqACP_{\Theta}^+$ are same, including the same operators and the same axiom systems, but different atomic action meanings (semantics).

So, we get $\langle z, \varrho \rangle \xrightarrow{q} \langle u, \varrho \rangle \Rightarrow pqACP_{\Theta}^+ \vdash z = u$, as desired. □

7. Conclusions

With an assumption of closed quantum systems, we unify quantum and classical computing under the framework of probabilistic process algebra, in which, quantum and classical computing have the same equational logic and the same structured operational semantics based on quantum configuration. Besides quantum computing mechanism in closed systems – unitary operator, quantum measurement and quantum entanglement, many computation properties are also involved, including sequential composition, alternative composition, probabilistic choice, recursion (projection), parallelism and communication, and abstraction. As a ACP-like process algebra, it inherits the advantages of ACP, such as modularity, which makes it can be extended easily and elegantly. Our work can be used widely in verification for quantum and classical computing mixed systems, such as most quantum communication protocol.

Note that, our work strongly relies on the progress of ACP-like probabilistic process algebra. For ACP-like probabilistic process algebra still can not solve a mixture of alternative composition, probabilistic choice and abstraction. This shortcoming is also one of our future directions.

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